Homogenization via unfolding in domains separated by the thin layer of the thin beams

Georges Griso, Anastasia Migunova, Julia Orlik

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Abstract

We consider a thin heterogeneous layer consisted of the thin beams (of radius r) and we study the limit behaviour of this problem as the periodicity ε, the thickness δ and the radius r of the beams tend to zero. The decomposition of the displacement field in the beams developed in [1] is used, which allows to obtain a priori estimates. Two types of the unfolding operators are introduced to deal with the different parts of the decomposition. In conclusion we obtain the limit problem together with the transmission conditions across the interface.

1 Introduction

In this paper a system of elasticity equations in the domains separated by a thin heterogeneous layer is considered. The layer is composed of periodically distributed vertical beams, which diameter and height tend to zero together with the period of the structure. The structure is clamped on the bottom. We consider the case of the isotropic linearized elasticity system.

The elasticity problems involving thin layers with periodic heterogeneous structure appear in many engineering and material sciences, where special constraints on stiffness or strength of technical textiles or composites are required, depending on the type of application. For example, drainages and protective wear, working for outer-plane compression, should provide certain stiffness and strength against external mechanical loading. Thin layers were considered in number of papers (see e.g. [9, 10, 11]). In particular, [9] deals with the layer composed of the holes scaled with additional small parameter; [10, 11] consider the case of the layer which stiffness has the same order as its thickness. as well The thin beams and their junction with the 3d structures were also studied in [1, 2, 3, 4]: [1] deals with the homogenization of a single thin body; in [2] the structure made of these bodies is considered. [3], [4] study the limit behavior of the structures composed of the rods in junction with a plate.

In our problem due to the combination of both models above we obtain 3 small parameters: the thickness δ of the layer (and the height of the beams at the same time), the radius r of the rods and the period of the layer ε. The first problem with this structure arises when we obtain the estimates on the displacements. To overcome this difficulty we used decomposition of the thin beams on the mean displacement and the rotation of the cross-section which was introduced in [1]. After deriving estimates on the components of the decomposition we obtain bounds for the minimizing sequence which depend on $\varepsilon, r\delta$. The result implies studying 3 critical cases with different ratios between small parameters. Two of them are considered in the present paper and lead to the same kind of the limit problem. The third one corresponds no longer to the thin beams but to the small inclusions and therefore is not studied in the present paper.

In order to obtain the limit problem the periodic unfolding method applied again to the components of the decomposition is used. Basic theory on the unfolding method can be found in [6]. In the present study we introduce two additional types of the unfolding operators in order to deal with the mean displacement and rotation which depend only on component x_3 and the warping which depends on all (x_1, x_2, x_3). In the limit we derive 3d elasticity problem for two domains separated by the interface with Robin-type condition on it. The value of this jump–condition is obtained from the solution of 1d beams problem.

The paper is organized as follows. In Section 2 geometry and weak and strong formulations of the problem are introduced. Section 3 presents decomposition of a single beam and the initial estimates. Section 4 is devoted to derivation of a priori estimates in all subdomains of $\Omega_{r,\varepsilon,\delta}$. In Section 5 the periodic unfolding operators are introduced and their properties are defined. Also the limit fields for the beams based on the estimates from the Section 4 are defined. Section 6 deals with passing to the limit and obtaining the variational formulation for the limit problem. In Section 7 the results are summarized: the strong formulation for the limit problem is given and the final result on the convergences of the solutions is introduced. Section 8 contains additional information. Section 9 includes subsidiary lemma which was used in the proofs.

2 The statement of the problem

2.1 Geometry

In the Euclidean space $\mathbb{R}^2$ let ω be a connected domain with Lipschitz boundary and let $L > 0$ be a fixed real number. Define the reference domains:

$$\begin{aligned} \Omega^- &= \omega \times (-L, 0), \\ \Omega^+ &= \omega \times (0, L), \\ \Sigma &= \omega \times \{0\}. \end{aligned}$$

Moreover, Ω (see Figure 1b) is defined by

$$\Omega = \Omega^+ \cup \Omega^- \cup \Sigma = \omega \times (-L, L). \tag{2.1}$$

For the domains corresponding to the structure with the layer of thicknes δ introduce the following notations:

$$\begin{aligned} \Omega^+_\delta &= \omega \times (\delta, L), \\ \Sigma^+_\delta &= \omega \times \{\delta\}. \end{aligned}$$

In order to describe the configuration of the layer, for any $(d, r) \in (0, +\infty)^2$ we define the rod $B_{r,d}$ by

$$B_{r,d} = D_r \times (0, d)$$

where $D_r = D(O, r)$ is the disc of center O and radius r.

The set of rods is

$$\Omega^i_{r,\varepsilon,\delta} = \bigcup_{\mathbf{i} \in \widehat{\Xi}_\varepsilon \times \{0\}} \left\{ x \in \mathbb{R}^3 \ \mid \ x \in \mathbf{i}\varepsilon + B_{r,\delta} \right\}, \tag{2.2}$$

where

$$\widehat{\Xi}_\varepsilon = \Big\{ \xi \in \mathbb{Z}^2 \ \mid \varepsilon(\xi + Y) \subset \omega \Big\}, \quad Y = \left(-\frac{1}{2}; \frac{1}{2} \right)^2. \tag{2.3}$$

Moreover, we set:

$$\widehat{\omega}_\varepsilon = \text{interior} \bigcup_{\mathbf{i} \in \widehat{\Xi}_\varepsilon} \varepsilon\big(\mathbf{i} + \overline{Y}\big). \tag{2.4}$$

The physical reference configuration (see Figure 1a) is defined by $\Omega_{r,\varepsilon,\delta}$:

$$\Omega_{r,\varepsilon,\delta} = \text{interior}\Big(\overline{\Omega^-} \cup \overline{\Omega^i_{r,\varepsilon,\delta}} \cup \overline{\Omega^+_\delta} \Big). \tag{2.5}$$

The structure is fixed on a part Γ with non null measure of the boundary $\partial\Omega^- \setminus \Sigma$.

We make the following assumptions:

$$r < \frac{\varepsilon}{2}, \qquad \frac{r}{\delta} \le C. \tag{2.6}$$

Here, the first assumption $(2.6)_1$ is a non penetration condition for the beams while with the second one, we want to eliminate the case $\frac{\delta}{r} \to 0$ which needs the use of tools for plates (see [1]).

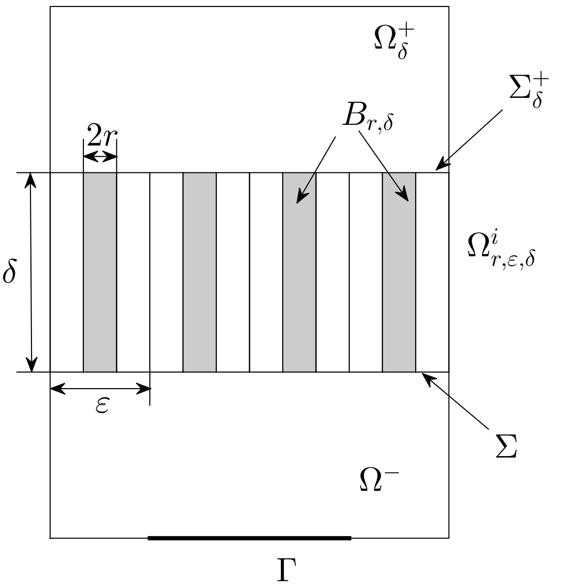
2.2 Strong formulation

Choose an isotropic material with Lamé constants λ^m, μ^m for the beams and another isotropic material with Lamé constants λ^b, μ^b for Ω^- and Ω^+_δ. Then we have the following values for the Poisson's coefficient of the material and Young's modulus:

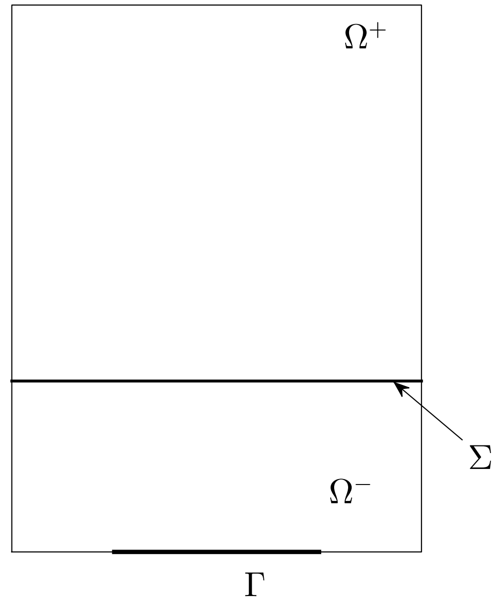
$$\begin{aligned} \nu^m &= \frac{\lambda^m}{2(\lambda^m + \mu^m)}, \quad & \nu^b &= \frac{\lambda^b}{2(\lambda^b + \mu^b)}, \\ E^m &= \frac{\mu^m(3\lambda^m + 2\mu^m)}{\lambda^m + \mu^m}, \quad & E^b &= \frac{\mu^b(3\lambda^b + 2\mu^b)}{\lambda^b + \mu^b}. \end{aligned}$$

The symmetric deformation field is defined by

$$(\nabla u)_S = \frac{\nabla u + \nabla^T u}{2}.$$



(a) The domain with the thin layer



(b) The limit problem

Figure 1: The reference configuration

The Cauchy stress tensor in $\Omega_{r,\varepsilon,\delta}$ is linked to $(\nabla u_{r,\varepsilon,\delta})_S$ through the standard Hooke's law:

$$\sigma_{r,\varepsilon,\delta} = \begin{cases} \lambda^b(\mathrm{Tr}\,(\nabla u_{r,\varepsilon,\delta})_S)I + 2\mu^b(\nabla u_{r,\varepsilon,\delta})_S & \text{in } \Omega^- \cup \Omega_\delta^+, \\ \lambda^m(\mathrm{Tr}\,(\nabla u_{r,\varepsilon,\delta})_S)I + 2\mu^m(\nabla u_{r,\varepsilon,\delta})_S & \text{in } \Omega^i_{r,\varepsilon,\delta}. \end{cases}$$

We consider the standard linear equations of elasticity in $\Omega_{r,\varepsilon,\delta}$. The unknown displacement $u_{r,\varepsilon,\delta} : \Omega_{r,\varepsilon,\delta} \to \mathbb{R}^3$ satisfies the following problem:

$$\begin{cases} \nabla \cdot \sigma_{r,\varepsilon,\delta} = -f_{r,\varepsilon,\delta} & \text{in } \Omega_{r,\varepsilon,\delta}, \\ u_{r,\varepsilon,\delta} = 0 & \text{on } \Gamma, \\ \sigma_{r,\varepsilon,\delta} \cdot \nu = 0 & \text{on } \partial\Omega_{r,\varepsilon,\delta} \setminus \Gamma. \end{cases} \tag{2.7}$$

2.3 Weak formulation

If $\mathbb{V}_{r,\varepsilon,\delta}$ denotes the space

$$\mathbb{V}_{r,\varepsilon,\delta} = \left\{ v \in H^1(\Omega_{r,\varepsilon,\delta}, \mathbb{R}^3) \mid v = 0 \text{ on } \Gamma \right\},$$

the variational formulation of (2.7) is

$$\begin{cases} \text{Find } u_{r,\varepsilon,\delta} \in \mathbb{V}_{r,\varepsilon,\delta}, \\ \int_{\Omega_{r,\varepsilon,\delta}} \sigma_{r,\varepsilon,\delta} : (\nabla\varphi)_S dx = \int_{\Omega_{r,\varepsilon,\delta}} f_{r,\varepsilon,\delta}\varphi dx, \quad \forall \varphi \in \mathbb{V}_{r,\varepsilon,\delta}. \end{cases} \tag{2.8}$$

We equip space $\mathbb{V}_{r,\varepsilon,\delta}$ with the following norm:

$$\|u\|_V = \|(\nabla u)_S\|_{L^2(\Omega_{r,\varepsilon,\delta})}.$$

It follows from the 3D-Korn inequality for domain Ω^-:

$$\|u\|_{H^1(\Omega^-)} \leq C\|(\nabla u)_S\|_{L^2(\Omega^-)}. \tag{2.9}$$

3 Decomposition of the displacements in $\Omega^i_{r,\varepsilon,\delta}$

3.1 Displacement of a single beam. Preliminary estimates

To obtain a priori estimates on $u_{r,\varepsilon,\delta}$ and $(\nabla u_{r,\varepsilon,\delta})_S$ we will need Korn's inequalities for this type of domain. However, for a multi-structure like this, it is not convenient to estimate the constant in a Korn's type inequality, because the order of each component of the displacement field may be very different. To overcome this difficulty, we will use a decomposition for the displacements of beams. A displacement of the beam $B_{r,d}$ is decomposed as the sum of three fields, the first one stands for the displacement of the center line, the second stands for the rotations of the cross sections and the last one is the warping, it takes into account the deformations of the cross sections.

We recall the definition of the elementary displacement from [1].

Definition 3.1. *The elementary displacement U_e, associated to $u \in L^1(B_{r,d}, \mathbb{R}^3)$, is given by*

$$U_e(x_1, x_2, x_3) = \mathcal{U}(x_3) + \mathcal{R}(x_3) \wedge (x_1 e_1 + x_2 e_2), \quad \textit{for a.e.} \quad x = (x_1, x_2, x_3) \in B_{r,d}, \tag{3.1}$$

where

$$\begin{cases} \mathcal{U} = \dfrac{1}{\pi r^2} \displaystyle\int_{D_r} u(x_1, x_2, \cdot) dx_1 dx_2, \\ \mathcal{R}_3 = \dfrac{1}{(I_1 + I_2) r^4} \displaystyle\int_{D_r} \big(x_1 u_2(x_1, x_2, \cdot) - x_2 u_1(x_1, x_2, \cdot)\big)\ dx_1 dx_2, \\ \mathcal{R}_\alpha = \dfrac{(-1)^{3-\alpha}}{I_{3-\alpha} r^4} \displaystyle\int_{D_r} x_{3-\alpha} u_3(x_1, x_2, \cdot)\ dx_1 dx_2, \quad I_\alpha = \displaystyle\int_{D_1} x_\alpha^2 dx_1 dx_2 = \dfrac{\pi}{4}. \end{cases} \tag{3.2}$$

We write

$$\bar{u} = u - U_e. \tag{3.3}$$

The displacement $\bar{u}$ is the warping.

The following theorem is proved in [1].

Theorem 3.1. *Let u be in $H^1(B_{r,d}; \mathbb{R}^3)$ and $u = U_e + \bar{u}$ the decomposition of u given by* (3.1)–(3.3)*. There exists a constant C independent of d and r such that the following estimates hold:*

$$\begin{aligned} & \|\bar{u}\|_{L^2(B_{r,d})} \le Cr \|(\nabla u)_S\|_{L^2(B_{r,d})}, \qquad \|\nabla \bar{u}\|_{L^2(B_{r,d})} \le C \|(\nabla u)_S\|_{L^2(B_{r,d})}, \\ & \Big\| \frac{d\mathcal{R}}{dx_3} \Big\|_{L^2(0,d)} \le \frac{C}{r^2} \|(\nabla u)_S\|_{L^2(B_{r,d})}, \\ & \Big\| \frac{d\mathcal{U}}{dx_3} - \mathcal{R} \wedge e_3 \Big\|_{L^2(0,d)} \le \frac{C}{r} \|(\nabla u)_S\|_{L^2(B_{r,d})}. \end{aligned} \tag{3.4}$$

We set

$$Y_\varepsilon = \varepsilon Y, \quad V_\varepsilon = Y_\varepsilon \times (-\varepsilon, 0), \quad B'_{r,\varepsilon} = D_r \times (-\varepsilon, 0), \quad V'_{r,\varepsilon,\delta} = V_\varepsilon \cup D_r \times (-\varepsilon, \delta).$$

Lemma 3.1. *Let u be in $H^1(V'_{r,\varepsilon,\delta}, \mathbb{R}^3)$ and $u = U_e + \bar{u}$ the decomposition of the restriction of u to the rod $B'_{r,\varepsilon}$ given by* (3.1)–(3.3)*. There exists a constant C independent of δ, ε and r such that the following estimates hold:*

$$\begin{aligned} & |\mathcal{R}(0)|^2 \le \frac{C}{r^3} \|\nabla u\|^2_{L^2(V_\varepsilon)}, \\ & \|\mathcal{R}\|^2_{L^2(0,\delta)} \le C\frac{\delta}{r^3} \|\nabla u\|^2_{L^2(V_\varepsilon)} + C\frac{\delta^2}{r^4} \|(\nabla u)_S\|^2_{L^2(B_{r,\delta})}, \\ & \Big\| \frac{d\mathcal{U}_\alpha}{dx_3} \Big\|^2_{L^2(0,\delta)} \le C\frac{\delta}{r^3} \|\nabla u\|^2_{L^2(V_\varepsilon)} + C\frac{\delta^2}{r^4} \|(\nabla u)_S\|^2_{L^2(B_{r,\delta})}, \\ & \|\mathcal{U}_3 - \mathcal{U}_3(0)\|^2_{L^2(0,\delta)} \le C\frac{\delta^2}{r^2} \|(\nabla u)_S\|^2_{L^2(B_{r,\delta})}, \\ & \|\mathcal{U}_\alpha - \mathcal{U}_\alpha(0)\|^2_{L^2(0,\delta)} \le C\frac{\delta^3}{r^3} \|\nabla u\|^2_{L^2(V_\varepsilon)} + C\frac{\delta^4}{r^4} \|(\nabla u)_S\|^2_{L^2(B_{r,\delta})}, \\ & \|u(\cdot,\cdot,0) - \mathcal{U}(0)\|^2_{L^2(Y_\varepsilon)} \le C\varepsilon \|\nabla u\|^2_{L^2(V_\varepsilon)} + C\frac{\varepsilon^2}{r} \|(\nabla u)_S\|^2_{L^2(V_\varepsilon)}. \end{aligned} \tag{3.5}$$

Proof. Applying the 2D-Poincaré-Wirtinger's inequality we obtain the following estimate:

$$\|u - \mathcal{U}\|_{L^2(B'_{r,\varepsilon})} \le Cr \|\nabla u\|_{L^2(B'_{r,\varepsilon})} \tag{3.6}$$

The constant does not depend on r and ε.

Step 1. Estimate of $\mathcal{R}(0)$.

Recalling the definition of $\mathcal{R}$ from (3.2) and since $\displaystyle\int_{D_r} x_1 dx_1 dx_2 = \int_{D_r} x_2 dx_1 dx_2 = 0$, we can write

$$\forall x_3 \in [-\varepsilon, 0], \qquad \mathcal{R}_1(x_3) = \frac{1}{I_2 r^4} \int_{D_r} x_2 \big(u_3(x) - \mathcal{U}_3(x_3)\big) dx_1 dx_2.$$

By Cauchy's inequality

$$\forall x_3 \in [-\varepsilon, 0], \qquad |\mathcal{R}_1(x_3)|^2 \leq \frac{1}{I_2^2 r^8} \int_{D_r} x^2 dx_1 dx_2 \times \int_{D_r} (u_3(x) - \mathcal{U}_3(x_3))^2 dx_1 dx_2$$
$$\leq \frac{C}{r^4} \int_{D_r} (u_3(x) - \mathcal{U}_3(x_3))^2 dx_1 dx_2.$$

Integrating with respect to x_3 gives

$$\int_{-\varepsilon}^{0} |\mathcal{R}_1(x_3)|^2 dx_3 \leq \frac{C}{r^4} \int_{B_{r,\varepsilon}} (u(x) - \mathcal{U}(x_3))^2 dx.$$

Using (3.6) we can write

$$\|\mathcal{R}_1\|_{L^2(-\varepsilon,0)} \leq \frac{C}{r} \|\nabla u\|_{L^2(B'_{r,\varepsilon})}. \tag{3.7}$$

The derivative of $\mathcal{R}_1$ is equal to $\displaystyle\frac{d\mathcal{R}_1}{dx_3}(x_3) = \frac{1}{I_2 r^4} \int_{D_r} x_2 \frac{\partial u_3(x)}{\partial x_3} dx_1 dx_2$ for a.e. $x_3 \in (-\varepsilon, 0)$. Then proceeding as above we obtain for a.e. $x_3 \in (-\varepsilon, 0)$

$$\left| \frac{d\mathcal{R}_1}{dx_3}(x_3) \right|^2 \leq \frac{C}{r^4} \int_{D_r} \left| \frac{\partial u_3(x)}{\partial x_3} \right|^2 dx_1 dx_2.$$

Hence

$$\left\| \frac{d\mathcal{R}_1}{dx_3} \right\|_{L^2(-\varepsilon,0)} \leq \frac{C}{r^2} \left\| \frac{\partial u_3}{\partial x_3} \right\|_{L^2(B'_{r,\varepsilon})} \leq \frac{C}{r^2} \|\nabla u\|_{L^2(B'_{r,\varepsilon})}. \tag{3.8}$$

We recall the following classical estimates for $\phi \in H^1(-a, 0)$ $(a > 0)$

$$\begin{aligned} |\phi(0)|^2 &\leq \frac{2}{a} \|\phi\|^2_{L^2(-a,0)} + \frac{a}{2} \|\phi'\|^2_{L^2(-a,0)}, \\ \|\phi\|^2_{L^2(-a,0)} &\leq 2a|\phi(0)|^2 + a^2 \|\phi'\|^2_{L^2(-a,0)}. \end{aligned} \tag{3.9}$$

Due to (3.7)-(3.8), $(3.9)_1$ with $a = r$ and since $\varepsilon > r$ that gives for $\mathcal{R}_1(0)$

$$|\mathcal{R}_1(0)|^2 \leq \frac{C}{r^3} \|\nabla u\|^2_{L^2(B'_{r,\varepsilon})}.$$

The estimates for $\mathcal{R}_2(0)$, $\mathcal{R}_3(0)$ are obtained in the same way. Hence we get $(3.5)_1$.

Step 2. Estimate of $\|\mathcal{R}\|_{L^2(0,\delta)}$.

The Poincaré's inequality leads to

$$\|\mathcal{R} - \mathcal{R}(0)\|_{L^2(0,\delta)} \leq \delta \left\| \frac{d\mathcal{R}}{dx_3} \right\|_{L^2(0,\delta)}.$$

From $(3.4)_3$, $(3.9)_2$ and $(3.5)_1$ we get

$$\|\mathcal{R}\|^2_{L^2(0,\delta)} \leq 2\delta|\mathcal{R}(0)|^2 + \delta^2 \left\| \frac{d\mathcal{R}}{dx_3} \right\|^2_{L^2(0,\delta)} \leq C\frac{\delta}{r^3} \|\nabla u\|^2_{L^2(B'_{r,\varepsilon})} + C\frac{\delta^2}{r^4} \|(\nabla u)_S\|^2_{L^2(B_{r,\delta})}. \tag{3.10}$$

Hence $(3.5)_2$ is proved.

Step 3. Estimate of $\mathcal{U} - \mathcal{U}(0)$.

Applying inequality $(3.4)_4$ from Theorem 3.1 the following estimates on $\mathcal{U}$ hold:

$$\begin{aligned} \left\| \frac{d\mathcal{U}_3}{dx_3} \right\|_{L^2(0,\delta)} &\leq \frac{C}{r} \|(\nabla u)_S\|_{L^2(B_{r,\delta})}, \\ \left\| \frac{d\mathcal{U}_\alpha}{dx_3} \right\|_{L^2(0,\delta)} &\leq \|\mathcal{R}\|_{L^2(0,\delta)} + \frac{C}{r} \|(\nabla u)_S\|_{L^2(B_{r,\delta})}. \end{aligned} \tag{3.11}$$

Combining $(3.11)_2$ with (3.10) gives

$$\left\| \frac{d\mathcal{U}_\alpha}{dx_3} \right\|^2_{L^2(0,\delta)} \leq C\frac{\delta}{r^3} \|\nabla u\|^2_{L^2(B'_{r,\varepsilon})} + C\frac{\delta^2}{r^4} \|(\nabla u)_S\|^2_{L^2(B_{r,\delta})} + \frac{C}{r^2} \|(\nabla u)_S\|^2_{L^2(B_{r,\delta})}.$$

Taking into account the assumption $(2.6)_2$, we obtain $(3.5)_3$. Then by $(3.5)_3$, $(3.11)_1$ and the Poincaré's inequality $(3.5)_4$, $(3.5)_5$ follow.

Step 4. We prove the estimate $(3.5)_6$.

By Korn inequality there exists rigid displacement $\mathbf{r}$

$$\begin{aligned}
&\mathbf{r}(x) = \mathbf{a} + \mathbf{b} \wedge \left(x + \frac{\varepsilon}{2}e_3\right),\\
&\mathbf{a} = \frac{1}{\varepsilon^3}\int_{V_\varepsilon} u(x)\,dx,\\
&\mathbf{b} = \frac{6}{\varepsilon^5}\int_{V_\varepsilon} \left(x + \frac{\varepsilon}{2}e_3\right) \wedge u(x)\,dx.
\end{aligned}$$

such that

$$\begin{aligned}
\|u - \mathbf{r}\|_{L^2(V_\varepsilon)} &\le C\varepsilon\|(\nabla u)_S\|_{L^2(V_\varepsilon)},\\
\|\nabla(u - \mathbf{r})\|_{L^2(V_\varepsilon)} &\le C\|(\nabla u)_S\|_{L^2(V_\varepsilon)}.
\end{aligned} \tag{3.12}$$

Besides by Poincaré-Wirtinger inequality we have

$$\|u - \mathbf{a}\|_{L^2(V_\varepsilon)} \le C\varepsilon\|\nabla u\|_{L^2(V_\varepsilon)}. \tag{3.13}$$

The Sobolev embedding theorems give ($V = Y \times (-1, 0)$)

$$\|\varphi\|_{L^4(Y)} \le C\|\varphi\|_{H^{1/2}(Y)} \le C\left(\|\varphi\|_{L^2(V)} + \|\nabla\varphi\|_{L^2(V)}\right), \quad \forall\varphi \in H^1(V).$$

By change of variables we obtain

$$\|\varphi\|_{L^4(Y_\varepsilon)} \le C\left(\frac{1}{\varepsilon}\|\varphi\|_{L^2(V_\varepsilon)} + \varepsilon\|\nabla\varphi\|_{L^2(V_\varepsilon)}\right), \quad \forall\varphi \in H^1(V_\varepsilon).$$

Therefore, (3.12) and the above inequality lead to

$$\|u - \mathbf{r}\|_{L^4(Y_\varepsilon)} \le C\|(\nabla u)_S\|_{L^2(V_\varepsilon)}. \tag{3.14}$$

From the identity

$$\frac{1}{\pi r^2}\int_{D_r} (u(x', 0) - \mathbf{r}(x', 0))dx' = \mathcal{U}(0) - \mathbf{a} - \mathbf{b} \wedge \frac{\varepsilon}{2}e_3,$$

estimate (3.14) and Hölder inequality we get

$$\left|\mathcal{U}(0) - \mathbf{a} - \mathbf{b} \wedge \frac{\varepsilon}{2}e_3\right| \le \frac{1}{\pi r^2}\left(\int_{D_r} 1^{4/3}dx'\right)^{3/4}\left(\int_{D_r} |u(x', 0) - \mathbf{r}(x', 0)|^4 dx'\right)^{1/4} \le \frac{C}{r^{1/2}}\|(\nabla u)_S\|_{L^2(V_\varepsilon)}. \tag{3.15}$$

From Cauchy-Schwarz inequality and taking into account (3.13), we derive

$$\begin{aligned}
|\mathbf{b}| &\le \frac{C}{\varepsilon^5}\left(\int_{V_\varepsilon}\left|x + \frac{\varepsilon}{2}e_3\right|^2 dx\right)^{1/2}\left(\int_{V_\varepsilon}|u(x) - \mathbf{a}|^2 dx\right)^{1/2}\\
&\le \frac{C}{\varepsilon^5}\cdot\varepsilon\cdot\varepsilon^{3/2}\|u - \mathbf{a}\|_{L^2(V_\varepsilon)} \le \frac{C}{\varepsilon^{5/2}}\varepsilon\|\nabla u\|_{L^2(V_\varepsilon)} \le \frac{C}{\varepsilon^{3/2}}\|\nabla u\|_{L^2(V_\varepsilon)}.
\end{aligned} \tag{3.16}$$

Using (3.15) and (3.16) we obtain

$$|\mathcal{U}(0) - \mathbf{a}| \le \left|\mathcal{U}(0) - \mathbf{a} - \mathbf{b} \wedge \frac{\varepsilon}{2}e_3\right| + \left|\mathbf{b} \wedge \frac{\varepsilon}{2}e_3\right| \le \frac{C}{r^{1/2}}\|(\nabla u)_S\|_{L^2(V_\varepsilon)} + \frac{C}{\varepsilon^{1/2}}\|\nabla u\|_{L^2(V_\varepsilon)}. \tag{3.17}$$

Estimates (3.9) and (3.13) yield

$$\|u(\cdot, \cdot, 0) - \mathbf{a}\|^2_{L^2(Y_\varepsilon)} \le C\varepsilon\|\nabla u\|^2_{L^2(V_\varepsilon)}. \tag{3.18}$$

Combining (3.17), (3.18) gives

$$\begin{aligned}
\|u(\cdot, \cdot, 0) - \mathcal{U}(0)\|^2_{L^2(Y_\varepsilon)} &\le C(\|u(\cdot, \cdot, 0) - \mathbf{a}\|^2_{L^2(Y_\varepsilon)} + \|\mathcal{U}(0) - \mathbf{a}\|^2_{L^2(Y_\varepsilon)})\\
&\le C\varepsilon\|\nabla u\|^2_{L^2(V_\varepsilon)} + C\frac{\varepsilon^2}{r}\|(\nabla u)_S\|^2_{L^2(V_\varepsilon)} + C\varepsilon\|\nabla u\|^2_{L^2(V_\varepsilon)}\\
&\le C\varepsilon\|\nabla u\|^2_{L^2(V_\varepsilon)} + C\frac{\varepsilon^2}{r}\|(\nabla u)_S\|^2_{L^2(V_\varepsilon)}.
\end{aligned}$$

Hence we get $(3.5)_6$. $\square$

4 A priori estimates

In this section all the constants do not depend on ε, δ and r. We denote $x' = (x_1, x_2)$ the running point of $\mathbb{R}^2$.

4.1 Decomposition of the displacements in $\Omega^i_{r,\varepsilon,\delta}$

We decompose the displacement $u \in \mathbb{V}_{r,\varepsilon,\delta}$ in each beam $\varepsilon\mathbf{i} + B_{r,\delta}$, $\mathbf{i} \in \widehat{\Xi}_\varepsilon \times \{0\}$ as in the Definition 3.1. The components of the elementary displacement are denoted $\mathcal{U}_\xi$, $\mathcal{R}_\xi$, where $\xi = \Big[\dfrac{x'}{\varepsilon}\Big]_Y$.

Now we define the fields $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{R}}$ and $\widetilde{\bar{u}}$ for a.e. $x \in B_{r,\delta}, s \in \omega$ by

$$\widetilde{\mathcal{U}}(s_1, s_2, x_3) = \begin{cases} \mathcal{U}_\xi(x_3), & \text{if } \xi = \Big[\dfrac{s}{\varepsilon}\Big] \in \widehat{\Xi}_\varepsilon \\ 0, & \text{if } \xi \notin \widehat{\Xi}_\varepsilon \end{cases}, \qquad \widetilde{\mathcal{R}}(s_1, s_2, x_3) = \begin{cases} \mathcal{R}_\xi(x_3), & \text{if } \xi = \Big[\dfrac{s}{\varepsilon}\Big] \in \widehat{\Xi}_\varepsilon \\ 0, & \text{if } \xi \notin \widehat{\Xi}_\varepsilon \end{cases},$$

$$\widetilde{\bar{u}}(s_1, s_2, x) = \begin{cases} \bar{u}_\xi(x), & \text{if } \xi = \Big[\dfrac{s}{\varepsilon}\Big] \in \widehat{\Xi}_\varepsilon \\ 0, & \text{if } \xi \notin \widehat{\Xi}_\varepsilon \end{cases}.$$

We have

$$\widetilde{\mathcal{U}}, \widetilde{\mathcal{R}} \in L^2(\omega, H^1((0,\delta), \mathbb{R}^3)), \quad \widetilde{\bar{u}} \in L^2(\omega, H^1(B_{r,\delta}, \mathbb{R}^3)).$$

Moreover,

$$\|\widetilde{\mathcal{U}}\|^2_{L^2(\omega\times(0,\delta))} = \varepsilon^2 \sum_{\xi\in\widehat{\Xi}_\varepsilon} \|\mathcal{U}_\xi\|^2_{L^2(0,\delta)}, \quad \|\widetilde{\mathcal{R}}\|^2_{L^2(\omega\times(0,\delta))} = \varepsilon^2 \sum_{\xi\in\widehat{\Xi}_\varepsilon} \|\mathcal{R}_\xi\|^2_{L^2(0,\delta)},$$
$$\|\widetilde{\bar{u}}\|^2_{L^2(\omega\times B_{r,\delta})} = \varepsilon^2 \sum_{\xi\in\widehat{\Xi}_\varepsilon} \|\bar{u}_\xi\|^2_{L^2(B_{r,\delta})}.$$

As a consequence of the Theorem 3.1 and Lemma 3.1 we get

Lemma 4.1. *Let u be in $\mathbb{V}_{r,\varepsilon,\delta}$. The following estimates hold:*

$$\begin{aligned}
&\Big\|\frac{\partial \widetilde{\mathcal{R}}}{\partial x_3}\Big\|_{L^2(\omega\times(0,\delta))} \le C\frac{\varepsilon}{r^2}\|u\|_V,\\
&\Big\|\frac{\partial \widetilde{\mathcal{U}}}{\partial x_3} - \widetilde{\mathcal{R}}\wedge e_3\Big\|_{L^2(\omega\times(0,\delta))} \le C\frac{\varepsilon}{r}\|u\|_V,\\
&\|\nabla_x \widetilde{\bar{u}}\|_{L^2(\omega\times B_{r,\delta})} \le C\varepsilon\|u\|_V,\\
&\|\widetilde{\bar{u}}\|_{L^2(\omega\times B_{r,\delta})} \le C\varepsilon r\|u\|_V,\\
&\|\widetilde{\mathcal{R}}\|_{L^2(\omega\times(0,\delta))} \le C\frac{\varepsilon\delta}{r^2}\|u\|_V,\\
&\Big\|\frac{\partial \widetilde{\mathcal{U}}_\alpha}{\partial x_3}\Big\|_{L^2(\omega\times(0,\delta))} \le C\frac{\varepsilon\delta}{r^2}\|u\|_V.
\end{aligned} \tag{4.1}$$

Moreover,

$$\begin{aligned}
&\|\widetilde{\mathcal{R}}(\cdot,\cdot,0)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\frac{\varepsilon^2}{r^3}\|u\|^2_V,\\
&\|\widetilde{\mathcal{R}}(\cdot,\cdot,\delta)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\frac{\varepsilon^2}{r^3}\|\nabla u\|^2_{L^2(\Omega^+_\delta)},\\
&\|\widetilde{\mathcal{U}}_3 - \widetilde{\mathcal{U}}_3(\cdot,\cdot,0)\|_{L^2(\omega\times(0,\delta))} \le C\frac{\delta\varepsilon}{r}\|u\|_V,\\
&\|\widetilde{\mathcal{U}}_\alpha - \widetilde{\mathcal{U}}_\alpha(\cdot,\cdot,0)\|_{L^2(\omega\times(0,\delta))} \le C\frac{\delta^2\varepsilon}{r^2}\|u\|_V, \quad \textit{where } \alpha = 1,2.
\end{aligned} \tag{4.2}$$

Proof. Estimates $(4.1)_1$ – $(4.1)_6$ follow directly from (2.9), $(3.4)_3$, $(3.4)_4$ and $(3.5)_2$–$(3.5)_3$ and estimates $(4.2)_1$ – $(4.2)_4$ are the consequences of the estimates in Lemma 3.1 and (2.9). $\square$

4.2 Estimates on the interface traces

Lemma 4.2. *There exists a constant C independent of ε, δ, r such that for any $u \in \mathbb{V}_{r,\varepsilon,\delta}$*

$$\|u(\cdot,\cdot,0) - \widetilde{\mathcal{U}}(\cdot,\cdot,0)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\frac{\varepsilon^2}{r}\|u\|^2_V, \tag{4.3}$$

$$\|u(\cdot,\cdot,\delta) - \widetilde{\mathcal{U}}(\cdot,\cdot,\delta)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\varepsilon\|\nabla u\|^2_{L^2(\Omega^+_\delta)} + C\frac{\varepsilon^2}{r}\|u\|^2_V. \tag{4.4}$$

Moreover,

$$\|u_\alpha(\cdot,\cdot,\delta) - u_\alpha(\cdot,\cdot,0)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\varepsilon\|\nabla u\|^2_{L^2(\Omega^+_\delta)} + C\frac{\varepsilon^2\delta^3}{r^4}\|u\|^2_V, \tag{4.5}$$

$$\|u_3(\cdot,\cdot,\delta) - u_3(\cdot,\cdot,0)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\varepsilon\|\nabla u\|^2_{L^2(\Omega^+_\delta)} + C\frac{\varepsilon^2\delta}{r^2}\|u\|^2_V. \tag{4.6}$$

Proof. Using $(3.5)_6$ and then summing all of the periodicity cells give

$$\|u(\cdot,\cdot,0) - \widetilde{\mathcal{U}}(\cdot,\cdot,0)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\varepsilon\|\nabla u\|^2_{L^2(\Omega^-)} + C\frac{\varepsilon^2}{r}\|u\|^2_V. \tag{4.7}$$

In the same way the following estimate is derived:

$$\|u(\cdot,\cdot,\delta) - \widetilde{\mathcal{U}}(\cdot,\cdot,\delta)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\varepsilon\|\nabla u\|^2_{L^2(\Omega^+_\delta)} + C\frac{\varepsilon^2}{r}\|u\|^2_V.$$

Applying $(4.1)_2$ we can write

$$\|\widetilde{\mathcal{U}}_3(\cdot,\cdot,\delta) - \widetilde{\mathcal{U}}_3(\cdot,\cdot,0)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le \delta\left\|\frac{\partial\widetilde{\mathcal{U}}_3}{\partial x_3}\right\|^2_{L^2(\widehat{\omega}_\varepsilon\times(0,\delta))} \le C\frac{\varepsilon^2\delta}{r^2}\|u\|^2_V. \tag{4.8}$$

From $(4.1)_6$ we have

$$\|\widetilde{\mathcal{U}}_\alpha(\cdot,\cdot,\delta) - \widetilde{\mathcal{U}}_\alpha(\cdot,\cdot,0)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\frac{\varepsilon^2\delta^2}{r^3}\|u\|^2_V + C\frac{\varepsilon^2\delta^3}{r^4}\|u\|^2_V \le C\frac{\varepsilon^2\delta^3}{r^4}\|u\|^2_V. \tag{4.9}$$

Using (4.9) and above estimates we obtain (4.5), (4.6). □

4.3 Estimates on the displacements in Ω^+_δ

Lemma 4.3. *There exists a constant C which does not depend on ε, r and δ, such that for any $u \in \mathbb{V}_{r,\varepsilon,\delta}$*

$$\|u_\alpha\|_{H^1(\Omega^+_\delta)} \le C\frac{\varepsilon\delta^{3/2}}{r^2}\|u\|_V + C\|u\|_V, \tag{4.10}$$

$$\|u_3\|_{H^1(\Omega^+_\delta)} \le C\frac{\varepsilon^{3/2}\delta^{3/2}}{r^2}\|u\|_V + C\|u\|_V, \tag{4.11}$$

where $\alpha = 1, 2$.

Proof. From the Korn's inequality and the trace theorem we derive

$$\begin{aligned} \|u\|_{L^2(\Sigma)} &\le C\|u\|_{H^1(\Omega^-)} \le C\|(\nabla u)_S\|_{L^2(\Omega^-)}, \\ \|u_i\|_{H^1(\Omega^+_\delta)} &\le C\Big(\|\nabla u_i\|_{L^2(\Omega^+_\delta)} + \|u_i\|_{L^2(\Sigma^+_\delta)}\Big), \qquad i \in \{1,2,3\}. \end{aligned} \tag{4.12}$$

We know that there exists a rigid displacement $\mathbf{r}$

$$\forall x \in \mathbb{R}^3, \qquad \mathbf{r}(x) = \mathbf{a} + \mathbf{b} \wedge (x - \delta e_3), \qquad \mathbf{a},\ \mathbf{b} \in \mathbb{R}^3,$$

such that

$$\|u - \mathbf{r}\|_{L^2(\Sigma^+_\delta)} \le C\|u - \mathbf{r}\|_{H^1(\Omega^+_\delta)} \le C\|(\nabla u)_S\|_{L^2(\Omega^+_\delta)}. \tag{4.13}$$

The constant does not depend on δ. Then, we get

$$\|(u - \mathbf{r})(\cdot,\cdot,\delta)\|_{L^2(\widehat{\omega}_\varepsilon)} \le \|u - \mathbf{r}\|_{L^2(\Sigma^+_\delta)} \le C\|(\nabla u)_S\|_{L^2(\Omega^+_\delta)}. \tag{4.14}$$

Using

$$\|u(\cdot,\cdot,0)\|_{L^2(\widehat{\omega}_\varepsilon)} \le \|u\|_{L^2(\Sigma)}, \tag{4.15}$$

from (4.5), (4.6) we obtain

$$\begin{aligned}
\|u_\alpha(\cdot,\cdot,\delta)\|_{L^2(\widehat{\omega}_\varepsilon)} &\le C\varepsilon^{1/2}\|\nabla u\|_{L^2(\Omega_\delta^+)} + C\frac{\varepsilon\delta^{3/2}}{r^2}\|u\|_V + C\|u\|_V,\\
\|u_3(\cdot,\cdot,\delta)\|_{L^2(\widehat{\omega}_\varepsilon)} &\le C\varepsilon^{1/2}\|\nabla u\|_{L^2(\Omega_\delta^+)} + C\frac{\varepsilon^{3/2}\delta^{1/2}}{r}\|u\|_V + C\|u\|_V.
\end{aligned} \tag{4.16}$$

Combining this with (4.14) gives

$$\begin{aligned}
\|\mathbf{r}_\alpha(\cdot,\cdot,\delta)\|_{L^2(\widehat{\omega}_\varepsilon)} &\le C\varepsilon^{1/2}\|\nabla u\|_{L^2(\Omega_\delta^+)} + C\frac{\varepsilon\delta^{3/2}}{r^2}\|u\|_V + C\|u\|_V,\\
\|\mathbf{r}_3(\cdot,\cdot,\delta)\|_{L^2(\widehat{\omega}_\varepsilon)} &\le C\varepsilon^{1/2}\|\nabla u\|_{L^2(\Omega_\delta^+)} + C\frac{\varepsilon\delta^{1/2}}{r}\|u\|_V + C\|u\|_V.
\end{aligned} \tag{4.17}$$

Therefore,

$$\begin{aligned}
|\mathbf{a_1}|+|\mathbf{a_2}|+|\mathbf{b_3}| &\le C\varepsilon^{1/2}\|\nabla u\|_{L^2(\Omega_\delta^+)} + C\frac{\varepsilon\delta^{3/2}}{r^2}\|u\|_V + C\|u\|_V,\\
|\mathbf{a_3}|+|\mathbf{b_1}|+|\mathbf{b_2}| &\le C\varepsilon^{1/2}\|\nabla u\|_{L^2(\Omega_\delta^+)} + C\frac{\varepsilon\delta^{1/2}}{r}\|u\|_V + C\|u\|_V.
\end{aligned}$$

These estimates together with (4.13) allow to obtain estimates on u_1, u_2, u_3. From this we have

$$\begin{aligned}
\|u_\alpha\|_{H^1(\Omega_\delta^+)} &\le C\varepsilon^{1/2}\|\nabla u\|_{L^2(\Omega_\delta^+)} + C\frac{\varepsilon\delta^{3/2}}{r^2}\|u\|_V + C\|u\|_V,\\
\|u_3\|_{H^1(\Omega_\delta^+)} &\le C\varepsilon^{1/2}\|\nabla u\|_{L^2(\Omega_\delta^+)} + C\frac{\varepsilon\delta^{1/2}}{r}\|u\|_V + C\|u\|_V.
\end{aligned} \tag{4.18}$$

Therefore,

$$\|\nabla u\|_{L^2(\Omega_\delta^+)} \le C\varepsilon^{1/2}\|\nabla u\|_{L^2(\Omega_\delta^+)} + C\frac{\varepsilon\delta^{3/2}}{r^2}\|u\|_V + C\|u\|_V.$$

For ε small enough the following holds true:

$$\|\nabla u\|_{L^2(\Omega_\delta^+)} \le C\frac{\varepsilon\delta^{3/2}}{r^2}\|u\|_V + C\|u\|_V.$$

Inserting this in (4.18) we derive (4.10)-(4.11). □

As a consequence of Lemma 4.3 and estimate (4.5), (4.6) can be replaced by

$$\|u_\alpha(\cdot,\cdot,\delta) - u_\alpha(\cdot,\cdot,0)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\frac{\varepsilon^2\delta^3}{r^4}\|u\|_V^2, \tag{4.19}$$

$$\|u_3(\cdot,\cdot,\delta) - u_3(\cdot,\cdot,0)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\frac{\varepsilon^3\delta^3}{r^4}\|u\|_V^2 + C\frac{\varepsilon^2\delta}{r^2}\|u\|_V^2. \tag{4.20}$$

4.4 Estimates for the set of beams $\Omega^i_{r,\varepsilon,\delta}$

Lemma 4.4. *There exists a constant C which does not depend on ε, r and δ, such that for any $u \in \mathbb{V}_{r,\varepsilon,\delta}$*

$$\begin{aligned}
\|\nabla u\|_{L^2(\Omega^i_{r,\varepsilon,\delta})} &\le C\frac{\delta}{r}\|u\|_V,\\
\|u_3\|_{L^2(\Omega^i_{r,\varepsilon,\delta})} &\le C\frac{r\delta^{1/2}}{\varepsilon}\|u\|_V,\\
\|u_\alpha\|_{L^2(\Omega^i_{r,\varepsilon,\delta})} &\le C\frac{r\delta^{1/2}}{\varepsilon}\left(1+\frac{\varepsilon\delta^{3/2}}{r^2}\right)\|u\|_V
\end{aligned} \tag{4.21}$$

where $\alpha = 1, 2$.

Proof. From the estimates in Theorem 3.1, $(3.5)_2$ and $(3.5)_3$ and after summation over all the beams, we get (we make use of the assumption $(2.6)_2$)

$$\|\nabla u\|^2_{L^2(\Omega^i_{r,\varepsilon,\delta})} \le C\left(\frac{\delta}{r}\|\nabla u\|^2_{L^2(\Omega^-)} + \frac{\delta^2}{r^2}\|(\nabla u)_S\|^2_{L^2(\Omega^i_{r,\varepsilon,\delta})}\right) \le C\frac{\delta^2}{r^2}\|u\|^2_V. \tag{4.22}$$

From (4.7) and $(4.12)_1$, it follows that

$$\sum_{\xi\in\Xi_\varepsilon} \varepsilon^2|\mathcal{U}_\xi(0)|^2 = \|\widetilde{\mathcal{U}}(\cdot,\cdot,0)\|^2_{L^2(\widehat{\omega}_\varepsilon)} \le C\frac{\varepsilon^2}{r}\|u\|^2_V + C\|u\|^2_V,$$
$$\sum_{\xi\in\Xi_\varepsilon} |\mathcal{U}_\xi(0)|^2 \le C\left(\frac{1}{r}+\frac{1}{\varepsilon^2}\right)\|u\|^2_V.$$

Using $(3.4)_4$, $(3.5)_3$, (3.9), we obtain

$$\begin{aligned}
&\sum_{\xi\in\Xi_\varepsilon} \|\mathcal{U}_{\xi,3}\|^2_{L^2(0,\delta)} \le C\left(\frac{\delta}{\varepsilon^2}+\frac{\delta^2}{r^2}\right)\|u\|^2_V,\\
&\sum_{\xi\in\Xi_\varepsilon} \|\mathcal{U}_{\xi,\alpha}\|^2_{L^2(0,\delta)} \le C\left(\frac{\delta}{\varepsilon^2}+\frac{\delta^4}{r^4}\right)\|u\|^2_V.
\end{aligned}\tag{4.23}$$

Additionally,

$$\sum_{\xi\in\Xi_\varepsilon} \|\bar{u}_\xi\|^2_{L^2(B_{r,\delta})} \le Cr^2\|(\nabla u)_S\|^2_{L^2(\Omega^i_{r,\varepsilon,\delta})} \le Cr^2\|u\|^2_V. \tag{4.24}$$

Then $(3.5)_2$, (4.23) and (4.24) give

$$\sum_{\xi\in\Xi_\varepsilon} \|u_{\xi,\alpha}\|^2_{L^2(B_{r,\delta})} \le C\left(\frac{r^2\delta}{\varepsilon^2}+\frac{\delta^4}{r^2}+\delta^2+r^2\right)\|u\|^2_V \le C\frac{r^2\delta}{\varepsilon^2}\left(1+\frac{\varepsilon^2\delta^3}{r^4}\right)\|u\|^2_V,$$
$$\sum_{\xi\in\Xi_\varepsilon} \|u_{\xi,3}\|^2_{L^2(B_{r,\delta})} \le C\left(\frac{r^2\delta}{\varepsilon^2}+\delta^2+r^2\right)\|u\|^2_V \le C\frac{r^2\delta}{\varepsilon^2}\left(1+\frac{\varepsilon^2\delta}{r^2}\right)\|u\|^2_V.$$

From the last inequalities we derive $(4.21)_2$ and $(4.21)_3$. □

4.5 The limit cases

In view of the estimates of Lemma 4.3 and in order that the lower and upper parts of our structure match, we must assume that

$$\frac{\varepsilon^2\delta^3}{r^4} \text{ is uniformly bounded from above.} \tag{4.25}$$

From now on, the parameters r, δ and ε are linked in this way

- $r=\kappa_0\varepsilon^{\eta_0}$, $\eta_0\ge 1$, $\kappa_0>0$, *if* $\eta_0=1$ *then* $\kappa_0\in(0,1/2)$ *(non penetration condition),*
- $\delta=\kappa_1\varepsilon^{\eta_1}$, $\kappa_1>0$ *and* $\eta_1\ge\eta_0$, *(in order to deal with the beams).*

The above assumption (4.25) *yields*

$$2+3\eta_1-4\eta_0\ge 0.$$

Hence we distinguish three important cases

- *(i)* $r=\kappa_0\varepsilon$, $\kappa_0\in(0,1/2)$ *and* $\delta=\kappa_1\varepsilon^{2/3}$, $\kappa_1>0$,
- *(ii)* $r=\kappa_0\varepsilon^{\eta_0}$, $\eta_0\in(1,2)$, $\kappa_0>0$ *and* $\delta=\kappa_1\varepsilon^{(4\eta_0-2)/3}$, $\kappa_1>0$,
- *(iii)* $r=\kappa_0\varepsilon^2$, $\kappa_0>0$ *and* $\delta=\kappa_1\varepsilon^2$, $\kappa_1>0$.

For the sake of simplicity, from now on we will use the following notations:

- Ω_ε instead of $\Omega_{r,\varepsilon,\delta}$,
- Ω^i_ε instead of $\Omega^i_{r,\varepsilon,\delta}$,
- Ω^+_ε instead of Ω^+_δ,

- σ_ε instead of $\sigma_{r,\varepsilon,\delta}$,
- u_ε instead of $u_{r,\varepsilon,\delta}$,
- f_ε instead of $f_{r,\varepsilon,\delta}$.

With assumption (4.25) we can rewrite some estimates obtained above. For any $u \in \mathbb{V}_{r,\varepsilon,\delta}$ we have

$$\|u\|_{L^2(\Omega^i_\varepsilon)} \le C\frac{r\delta^{1/2}}{\varepsilon}\|u\|_V, \tag{4.26}$$

$$\|u\|_{H^1(\Omega^+_\varepsilon)} \le C\|u\|_V. \tag{4.27}$$

The constants do not depend on ε, r and δ.

4.6 Force assumptions

We set

$$B_1 = D_1 \times (0,1).$$

To obtain estimates on u_ε we test (2.8) with $\varphi = u_\varepsilon$. We have

$$M_1\|u_\varepsilon\|^2_V \le \|f_\varepsilon\|_{L^2(\Omega_\varepsilon,\mathbb{R}^3)}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon,\mathbb{R}^3)}. \tag{4.28}$$

We consider the following assumption on the applied forces:

$$f_\varepsilon(x) = \begin{cases} \dfrac{\varepsilon^2}{r^2\delta} F^m\left(\varepsilon\left[\dfrac{x'}{\varepsilon}\right]_Y, \dfrac{\varepsilon}{r}\left\{\dfrac{x'}{\varepsilon}\right\}_Y, \dfrac{x_3}{\delta}\right) & \text{for a.e. } x \in \Omega^i_\varepsilon, \\ F(x) & \text{for a.e. } x \in \Omega^- \cup \Omega^+_\varepsilon, \end{cases} \tag{4.29}$$

where $F^m \in \mathcal{C}^0(\overline{\omega}, L^2(B_1,\mathbb{R}^3))$, $F \in L^2(\omega \times (-L,L),\mathbb{R}^3)$. Then,

$$\|f_\varepsilon\|_{L^2(\Omega^i_\varepsilon,\mathbb{R}^3)} \le \frac{\varepsilon}{r\delta^{1/2}}\|F^m\|_{L^\infty(\omega,L^2(B_1,\mathbb{R}^3))}.$$

Making use of the estimates (2.9), (4.26), (4.27) together with inequality (4.28) yield

$$\|u_\varepsilon\|_V \le C \tag{4.30}$$

The constant does not depend of r, ε and δ.

From now on, we only consider the cases (i) and (ii) introduced in Section 4.5.

5 The periodic unfolding operators

Definition 5.1. *For φ Lebesgue-measurable function on $\omega \times (0,\delta)$, the unfolding operator $\mathcal{T}_\varepsilon$ is defined as follows:*

$$\mathcal{T}_\varepsilon(\varphi)(s_1,s_2,X_3) = \begin{cases} \varphi(s_1,s_2,\delta X_3), & \text{for a.e. } (s_1,s_2,X_3) \in \widehat{\omega}_\varepsilon \times (0,1), \\ 0, & \text{for a.e. } (s_1,s_2,X_3) \in \Lambda_\varepsilon \times (0,1). \end{cases}$$

Definition 5.2. *For φ Lebesgue-measurable function on $\omega \times B_{r,\delta}$, the unfolding operator $\mathcal{T}'_\varepsilon$ is defined as follows:*

$$\mathcal{T}'_\varepsilon(\varphi)(s_1,s_2,X_1,X_2,X_3) = \begin{cases} \varphi(s_1,s_2,rX_1,rX_2,\delta X_3), & \text{for a.e. } (s_1,s_2,X_1,X_2,X_3) \in \widehat{\omega}_\varepsilon \times B_1, \\ 0, & \text{for a.e. } (s_1,s_2,X_1,X_2,X_3) \in \Lambda_\varepsilon \times B_1. \end{cases}$$

Observe that if φ is a Lebesgue-measurable function on $\omega \times (0,\delta)$ then $\mathcal{T}_\varepsilon(\varphi) = \mathcal{T}'_\varepsilon(\varphi)$.

Lemma 5.1. *(Properties of the operators $\mathcal{T}_\varepsilon$, $\mathcal{T}'_\varepsilon$)*

1. $\forall v, w \in L^2(\omega \times (0,\delta))$

$$\mathcal{T}_\varepsilon(vw) = \mathcal{T}_\varepsilon(v)\mathcal{T}_\varepsilon(w),$$

$\forall v, w \in L^2(\omega \times B_{r,\delta})$

$$\mathcal{T}'_\varepsilon(vw) = \mathcal{T}'_\varepsilon(v)\mathcal{T}'_\varepsilon(w).$$

2. $\forall u \in L^1(\omega \times (0,\delta))$

$$\delta \int_{\omega\times(0,1)} \mathcal{T}_\varepsilon(u)\, ds_1\, ds_2\, dX_3 = \int_{\widehat{\omega}_\varepsilon\times(0,\delta)} u\, ds_1\, ds_2\, dx_3,$$

$\forall u \in L^1(\omega \times B_{r,\delta})$

$$r^2\delta \int_{\omega\times B_1} \mathcal{T}_\varepsilon^{'}(u)\, ds_1\, ds_2\, dX_1\, dX_2\, dX_3 = \int_{\widehat{\omega}_\varepsilon\times B_{r,\delta}} u\, ds_1\, ds_2\, dx_1\, dx_2\, dx_3.$$

3. $\forall u \in L^2(\omega \times (0,\delta))$

$$\|\mathcal{T}_\varepsilon(u)\|_{L^2(\omega\times(0,1))} \le \frac{1}{\sqrt{\delta}}\|u\|_{L^2(\omega\times(0,\delta))},$$

$\forall u \in L^2(\omega \times B_{r,\delta})$

$$\|\mathcal{T}_\varepsilon^{'}(u)\|_{L^2(\omega\times B_1)} \le \frac{1}{r\sqrt{\delta}}\|u\|_{L^2(\omega\times B_{r,\delta})}.$$

4. Let u be in $L^2(\omega, H^1(0,\delta))$, a.e. in $\omega\times(0,1)$ we have

$$\delta\mathcal{T}_\varepsilon(\nabla_{x_3}u) = \nabla_{X_3}\mathcal{T}_\varepsilon(u).$$

Let u be in $L^2(\omega, H^1(B_{r,\delta}))$., a.e. in $\omega\times B_1$ we have

$$r\mathcal{T}_\varepsilon^{'}(\nabla_{x_\alpha}u) = \nabla_{X_\alpha}\mathcal{T}_\varepsilon^{'}(u), \quad \delta\mathcal{T}_\varepsilon^{'}(\nabla_{x_3}u) = \nabla_{X_3}\mathcal{T}_\varepsilon^{'}(u), \text{ where } \alpha = 1,2.$$

Proof. Properties 1-3 are obtained similarly as in the proof of Lemma 5.1 of [3].
Property 4 is the direct consequence chain rule formulae:

$$\frac{\partial(\mathcal{T}_\varepsilon^{'}(u))}{\partial X_\alpha} = r\mathcal{T}_\varepsilon^{'}\left(\frac{\partial u}{\partial x_\alpha}\right), \quad \alpha = 1,2,$$
$$\frac{\partial(\mathcal{T}_\varepsilon(u))}{\partial X_3} = \delta\mathcal{T}_\varepsilon\left(\frac{\partial u}{\partial x_3}\right), \quad \frac{\partial(\mathcal{T}_\varepsilon^{'}(u))}{\partial X_3} = \delta\mathcal{T}_\varepsilon^{'}\left(\frac{\partial u}{\partial x_3}\right).$$ □

5.1 The limit fields (Cases (i) and (ii))

From now on, $(u_\varepsilon)_\alpha$ will be denoted as $u_{\varepsilon,\alpha}$; the same notation will be used for the fields with values in $\mathbb{R}^2$ or $\mathbb{R}^3$.

From Lemmas 4.1 and 5.1 we obtain the following result.

Lemma 5.2. *There exists a constant C independent of ε, δ and r such that*

$$\|\mathcal{T}_\varepsilon(\widetilde{\mathcal{U}}_\varepsilon)\|_{L^2(\omega,H^1(0,1))} \le C, \tag{5.1}$$

$$\|\mathcal{T}_\varepsilon(\widetilde{\mathcal{U}}_{\varepsilon,3}) - \widetilde{\mathcal{U}}_{\varepsilon,3}(\cdot,\cdot,0)\|_{L^2(\omega,H^1(0,1))} \le C\frac{r}{\delta}, \tag{5.2}$$

$$\|\mathcal{T}_\varepsilon(\widetilde{\mathcal{R}}_\varepsilon)\|_{L^2(\omega,H^1(0,1))} \le \frac{C}{\delta}, \tag{5.3}$$

$$\Big\|\frac{\partial\mathcal{T}_\varepsilon(\widetilde{\mathcal{U}}_\varepsilon)}{\partial X_3} - \delta\mathcal{T}_\varepsilon(\widetilde{\mathcal{R}}_\varepsilon)\wedge e_3\Big\|_{L^2(\omega\times(0,1))} \le C\frac{r}{\delta}, \tag{5.4}$$

$$\|\mathcal{T}_\varepsilon^{'}(\widetilde{\widetilde{u}}_\varepsilon)\|_{L^2(\omega\times(0,1),H^1(D_1))} \le C\frac{r^2}{\delta^2}, \tag{5.5}$$

$$\left\|\frac{\partial\mathcal{T}_\varepsilon^{'}(\widetilde{\widetilde{u}}_\varepsilon)}{\partial X_3}\right\|_{L^2(\omega\times B_1)} \le C\frac{r}{\delta}. \tag{5.6}$$

Further we extend function u_ε defined on the domain Ω_ε^+ by reflection to the domain $\omega\times(\delta, L+\delta)$. The new function is denoted u_ε as before.

Proposition 5.1. *There exist a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, and $u^\pm \in H^1(\Omega^\pm,\mathbb{R}^3)$ with $u^- = 0$ on Γ, $\widetilde{\mathcal{R}} \in L^2(\omega, H_0^1((0,1),\mathbb{R}^3))$, $\widetilde{\mathcal{U}}_\alpha \in L^2(\omega, H^2(0,1))$, $\widetilde{\mathcal{U}}_3$, $\widetilde{\mathcal{U}}_3' \in L^2(\omega, H^1(0,1))$, $\widetilde{\widetilde{u}} \in L^2(\omega\times(0,1), H^1(D_1,\mathbb{R}^3))$ and $Z \in L^2(\omega\times(0,1),\mathbb{R}^3)$ such that*

$$u_\varepsilon \rightharpoonup u^- \quad \text{weakly in } H^1(\Omega^-), \text{ strongly in } L^2(\Omega^-), \tag{5.7}$$

$$u_\varepsilon(\cdot + \delta e_3) \rightharpoonup u^+ \quad \text{weakly in } H^1(\Omega^+), \text{ strongly in } L^2(\Omega^+), \tag{5.8}$$

$$\delta \mathcal{T}_\varepsilon(\widetilde{\mathcal{R}}_\varepsilon) \rightharpoonup \widetilde{\mathcal{R}} \quad \text{weakly in } L^2(\omega, H^1(0,1)), \text{ such that} \tag{5.9}$$

$$\widetilde{\mathcal{R}}(x',0) = \widetilde{\mathcal{R}}(x',1) = 0, \text{ for a.e. } x' \in \omega, \tag{5.10}$$

$$\frac{\delta}{r}\big(\mathcal{T}_\varepsilon(\widetilde{\mathcal{U}}_{\varepsilon,3}) - \widetilde{\mathcal{U}}_{\varepsilon,3}(\cdot,\cdot,0)\big) \rightharpoonup \widetilde{\mathcal{U}}_3' \quad \text{weakly in } L^2(\omega, H^1(0,1)), \tag{5.11}$$

$$\mathcal{T}_\varepsilon(\widetilde{\mathcal{U}}_{\varepsilon,3}) \rightharpoonup \widetilde{\mathcal{U}}_3 \quad \text{weakly in } L^2(\omega, H^1(0,1)), \text{ such that} \tag{5.12}$$

$$\widetilde{\mathcal{U}}_3(\cdot,\cdot,\cdot) = \widetilde{\mathcal{U}}_3(\cdot,\cdot,0) = u^-_{3\,|\Sigma} = \widetilde{\mathcal{U}}_3(\cdot,\cdot,1) = u^+_{3\,|\Sigma}, \text{ a.e. in } \omega \times (0,1), \tag{5.13}$$

$$\mathcal{T}_\varepsilon(\widetilde{\mathcal{U}}_{\varepsilon,\alpha}) \rightharpoonup \widetilde{\mathcal{U}}_\alpha \quad \text{weakly in } L^2(\omega, H^1(0,1)), \quad \text{for } \alpha = \overline{1,2}, \text{ such that} \tag{5.14}$$

$$\widetilde{\mathcal{U}}_\alpha(\cdot,\cdot,0) = u^-_{\alpha\,|\Sigma}, \quad \widetilde{\mathcal{U}}_\alpha(\cdot,\cdot,1) = u^+_{\alpha\,|\Sigma} \text{ a.e. in } \omega, \tag{5.15}$$

$$\frac{\partial \widetilde{\mathcal{U}}_\alpha}{\partial X_3}(\cdot,\cdot,0) = \frac{\partial \widetilde{\mathcal{U}}_\alpha}{\partial X_3}(\cdot,\cdot,1) = 0 \text{ a.e. in } \omega, \tag{5.16}$$

$$\frac{\partial \widetilde{\mathcal{U}}_1}{\partial X_3} = \widetilde{\mathcal{R}}_2, \quad \frac{\partial \widetilde{\mathcal{U}}_2}{\partial X_3} = -\widetilde{\mathcal{R}}_1 \text{ a.e. in } \omega \times (0,1), \tag{5.17}$$

$$\frac{\delta^2}{r^2}\mathcal{T}_\varepsilon'(\widetilde{\bar{u}}_\varepsilon) \rightharpoonup \widetilde{\bar{u}} \quad \text{weakly in } L^2(\omega \times (0,1), H^1(D_1)), \tag{5.18}$$

$$\frac{\delta}{r}\mathcal{T}_\varepsilon'(\widetilde{\bar{u}}_\varepsilon) \rightharpoonup 0 \quad \text{weakly in } L^2(\omega, H^1(B_1)), \tag{5.19}$$

$$\frac{\delta}{r}\left(\frac{\partial \mathcal{T}_\varepsilon(\widetilde{\mathcal{U}}_\varepsilon)}{\partial X_3} - \delta \mathcal{T}_\varepsilon(\widetilde{\mathcal{R}}_\varepsilon \wedge e_3)\right) \rightharpoonup Z \quad \text{weakly in } L^2(\omega \times (0,1)). \tag{5.20}$$

Proof. Convergences (5.7), (5.8), (5.9), (5.11), (5.12), (5.14),(5.18) and (5.20) follow from the estimate (4.30) and those in Lemma 5.2.
Equalities (5.10) are the consequences of $(4.2)_1$-$(4.2)_2$. To obtain (5.17) take into account that from (5.20) we have

$$\frac{\partial \widetilde{\mathcal{U}}}{\partial X_3} - \widetilde{\mathcal{R}} \wedge e_3 = \begin{pmatrix} \dfrac{\partial \widetilde{\mathcal{U}}_1}{\partial X_3} - \widetilde{\mathcal{R}}_2 \\ \dfrac{\partial \widetilde{\mathcal{U}}_2}{\partial X_3} + \widetilde{\mathcal{R}}_1 \\ \dfrac{\partial \widetilde{\mathcal{U}}_3}{\partial X_3} \end{pmatrix} = 0.$$

Then (5.10) yields (5.16). Equalities (5.13) are the consequences of $\dfrac{\partial \widetilde{\mathcal{U}}_3}{\partial X_3} = 0$ and the estimates (4.3), (4.4). Again due to (4.3), (4.4), we obtain

$$\widetilde{\mathcal{U}}_\alpha(x',0) = u^-_{\alpha\,|\Sigma}(x'), \qquad \widetilde{\mathcal{U}}_\alpha(x',1) = u^+_{\alpha\,|\Sigma}(x'), \qquad \text{for a.e. } x' \in \omega.$$

From Lemma 5.2 we have $\|\mathcal{T}_\varepsilon'(\widetilde{\bar{u}}_\varepsilon)\|_{L^2(\omega,H^1(B_1))} \le C\dfrac{r}{\delta}$ from which and (5.18) we deduce (5.19). $\square$

The strain tensor of the displacement u_ε is

$$\mathcal{T}_\varepsilon'\left((\widetilde{\nabla u_\varepsilon})_S\right)_{ij} = \mathcal{T}_\varepsilon'((\nabla \widetilde{\bar{u}}_\varepsilon)_S)_{ij}, \quad i,j = 1,2,$$

$$\mathcal{T}_\varepsilon'\left((\widetilde{\nabla u_\varepsilon})_S\right)_{13} = \frac{1}{2}\left(\left(\frac{1}{\delta}\frac{\partial \mathcal{T}_\varepsilon(\widetilde{\mathcal{U}}_{\varepsilon,1})}{\partial X_3} - \mathcal{T}_\varepsilon(\widetilde{\mathcal{R}}_{\varepsilon,2})\right) - \frac{r}{\delta}\frac{\partial \mathcal{T}_\varepsilon(\widetilde{\mathcal{R}}_{\varepsilon,3})}{\partial X_3}X_2\right) + \mathcal{T}_\varepsilon'((\nabla \widetilde{\bar{u}}_\varepsilon)_S)_{13},$$

$$\mathcal{T}_\varepsilon'\left((\widetilde{\nabla u_\varepsilon})_S\right)_{23} = \frac{1}{2}\left(\left(\frac{1}{\delta}\frac{\partial \mathcal{T}_\varepsilon(\widetilde{\mathcal{U}}_{\varepsilon,2})}{\partial X_3} + \mathcal{T}_\varepsilon(\widetilde{\mathcal{R}}_{\varepsilon,1})\right) + \frac{r}{\delta}\frac{\partial \mathcal{T}_\varepsilon(\widetilde{\mathcal{R}}_{\varepsilon,3})}{\partial X_3}X_1\right) + \mathcal{T}_\varepsilon'((\nabla \widetilde{\bar{u}}_\varepsilon)_S)_{23},$$

$$\mathcal{T}_\varepsilon'\left((\widetilde{\nabla u_\varepsilon})_S\right)_{33} = \frac{1}{\delta}\frac{\partial \mathcal{T}_\varepsilon(\widetilde{\mathcal{U}}_{\varepsilon,3})}{\partial X_3} + \frac{r}{\delta}\frac{\partial \mathcal{T}_\varepsilon(\widetilde{\mathcal{R}}_{\varepsilon,1})}{\partial X_3}X_2 - \frac{r}{\delta}\frac{\partial \mathcal{T}_\varepsilon(\widetilde{\mathcal{R}}_{\varepsilon,2})}{\partial X_3}X_1 + \mathcal{T}_\varepsilon'((\nabla \widetilde{\bar{u}}_\varepsilon)_S)_{33}.$$

Define the field $\widetilde{\overline{u}}' \in L^2(\omega \times (0,1), H^1(D_1,\mathbb{R}^3))$ by

$$\widetilde{\overline{u}}'_\alpha = \widetilde{\overline{u}}_\alpha, \qquad \widetilde{\overline{u}}'_3 = \widetilde{\overline{u}}_3 + X_1 Z_1 + X_2 Z_2.$$

Then

$$\frac{\partial \widetilde{\overline{u}}'_3}{\partial X_1} = \frac{\partial \widetilde{\overline{u}}_3}{\partial X_1} + Z_1, \qquad \frac{\partial \widetilde{\overline{u}}'_3}{\partial X_2} = \frac{\partial \widetilde{\overline{u}}_3}{\partial X_2} + Z_2.$$

As an immediate consequence of Proposition 5.1, we have

Lemma 5.3. *There exist a symmetric matrix field $X \in L^2(\omega \times B_1, \mathbb{R}^9)$ and a field $\widetilde{\overline{u}}' \in L^2(\omega \times (0,1), H^1(D_1,\mathbb{R}^3))$, such that*

$$\frac{\delta^2}{r}\mathcal{T}'_\varepsilon\left((\widetilde{\nabla u_\varepsilon})_S\right) \rightharpoonup X \qquad \text{weakly in } L^2(\omega \times B_1, \mathbb{R}^9),$$

where X is defined by

$$\begin{aligned}
X_{ij} &= \frac{1}{2}\left(\frac{\partial \widetilde{\overline{u}}'_i}{\partial X_j} + \frac{\partial \widetilde{\overline{u}}'_j}{\partial X_i}\right), \quad i,j = 1,2,\\
X_{13} = X_{31} &= \frac{1}{2}\left(\frac{\partial \widetilde{\overline{u}}'_3}{\partial X_1} - \frac{\partial \widetilde{\mathcal{R}}_3}{\partial X_3} X_2\right),\\
X_{23} = X_{32} &= \frac{1}{2}\left(\frac{\partial \widetilde{\overline{u}}'_3}{\partial X_2} + \frac{\partial \widetilde{\mathcal{R}}_3}{\partial X_3} X_1\right),\\
X_{33} &= \frac{\partial \widetilde{\mathcal{U}}'_3}{\partial X_3} - \frac{\partial^2 \widetilde{\mathcal{U}}_2}{\partial X_3^2} X_2 - \frac{\partial^2 \widetilde{\mathcal{U}}_1}{\partial X_3^2} X_1.
\end{aligned}$$

6 The limit problem

6.1 The equations for the domain Ω^i_ε

Denote by Θ the weak limit of the unfolded stress tensor $\frac{\delta^2}{r}\mathcal{T}'_\varepsilon(\sigma_\varepsilon)$ in $L^2(\omega \times B_1, \mathbb{R}^9)$:

$$\frac{\delta^2}{r}\mathcal{T}'_\varepsilon(\sigma_\varepsilon) \rightharpoonup \Theta, \qquad \text{weakly in } L^2(\omega \times B_1, \mathbb{R}^9).$$

Proceeding exactly as in Section 6.1 of [3] and Section 8.1 of [4], we first derive $\widetilde{\overline{u}}'$ and this gives

$$\begin{aligned}
\widetilde{\overline{u}}'_1 &= \nu^m\left(-X_1 \frac{\partial \widetilde{\mathcal{U}}'_3}{\partial X_3} + \frac{X_1^2 - X_2^2}{2}\frac{\partial^2 \widetilde{\mathcal{U}}_1}{\partial X_3^2} + X_1 X_2 \frac{\partial^2 \widetilde{\mathcal{U}}_2}{\partial X_3^2}\right),\\
\widetilde{\overline{u}}'_2 &= \nu^m\left(-X_2 \frac{\partial \widetilde{\mathcal{U}}'_3}{\partial X_3} + X_1 X_2 \frac{\partial^2 \widetilde{\mathcal{U}}_1}{\partial X_3^2} + \frac{X_2^2 - X_1^2}{2}\frac{\partial^2 \widetilde{\mathcal{U}}_2}{\partial X_3^2}\right).
\end{aligned}$$

Similarly, the same computations as in Section 6.1 of [3] lead to $\widetilde{\overline{u}}'_3 = 0$.

As a consequence from Lemma 5.3 we obtain

$$\begin{aligned}
&\Theta_{11} = \Theta_{22} = \Theta_{12} = 0,\\
&\Theta_{13} = -\mu^m X_2 \frac{\partial \widetilde{\mathcal{R}}_3}{\partial X_3}, \quad \Theta_{23} = \mu^m X_1 \frac{\partial \widetilde{\mathcal{R}}_3}{\partial X_3},\\
&\Theta_{33} = E^m\Big(\frac{\partial \widetilde{\mathcal{U}}'_3}{\partial X_3} - X_1 \frac{\partial^2 \widetilde{\mathcal{U}}_1}{\partial X_3^2} - X_2 \frac{\partial^2 \widetilde{\mathcal{U}}_2}{\partial X_3^2}\Big).
\end{aligned} \tag{6.1}$$

Proposition 6.1. $(\widetilde{\mathcal{U}}_1, \widetilde{\mathcal{U}}_2)$ *satisfy the variational formulation*

$$\frac{\pi\kappa_0^2}{4\kappa_1^4}E^m \int_0^1 \frac{\partial^2 \widetilde{\mathcal{U}}_\alpha}{\partial X_3^2}(x', X_3)\frac{d^2\varphi_\alpha}{dX_3^2}(X_3)\, dX_3 = \int_0^1 \widetilde{F}^m_\alpha(x', X_3)\varphi_\alpha(X_3)\, dX_3, \tag{6.2}$$
$$\forall \varphi_\alpha \in H_0^2(0,1), \qquad \textit{for a.e. } x' \in \omega$$

where

$$\widetilde{F}^m_\alpha(x', X_3) = \int_{D_1} F^m_\alpha(x', X) dX_1\, dX_2 \quad a.e. \ in\ \omega \times (0,1) \quad \alpha = 1, 2.$$

Furthermore $\widetilde{\mathcal{R}}_3 = 0$ *and there exists* $a \in L^2(\omega)$ *such that*

$$\widetilde{\mathcal{U}}'_3(x', X_3) = a(x') X_3 \qquad a.e.\ in\ \omega \times (0,1).$$

Proof. Step 1. Obtain the limit equations in Ω^i_ε.

We will use the following test function:

$$v_\varepsilon(x) = \frac{r}{\delta}\psi(\varepsilon\xi)\begin{pmatrix} \dfrac{\delta}{r}\varphi_1\left(\dfrac{x_3}{\delta}\right) - \dfrac{x_2 - \varepsilon\xi_2}{r}\varphi_4\left(\dfrac{x_3}{\delta}\right) \\ \dfrac{\delta}{r}\varphi_2\left(\dfrac{x_3}{\delta}\right) + \dfrac{x_1 - \varepsilon\xi_1}{r}\varphi_4\left(\dfrac{x_3}{\delta}\right) \\ \varphi_3\left(\dfrac{x_3}{\delta}\right) - \dfrac{x_1 - \varepsilon\xi_1}{r}\dfrac{d\varphi_1}{dX_3}\left(\dfrac{x_3}{\delta}\right) - \dfrac{x_2 - \varepsilon\xi_2}{r}\dfrac{d\varphi_2}{dX_3}\left(\dfrac{x_3}{\delta}\right) \end{pmatrix}, \qquad \xi = \left[\frac{x'}{\varepsilon}\right]_Y,$$

where $\psi \in C^\infty_c(\omega)$, φ_3 and $\varphi_4 \in H^1_0(0,1)$, φ_1 and $\varphi_2 \in H^2_0(0,1)$. Computation of the symmetric strain tensor gives

$$(\nabla v_\varepsilon)_S = \frac{r}{\delta^2}\psi(\varepsilon\xi)\begin{pmatrix} 0 & 0 & -\dfrac{1}{2}\dfrac{x_2 - \varepsilon\xi_2}{r}\dfrac{d\varphi_4}{dX_3} \\ \dots & 0 & \dfrac{1}{2}\dfrac{x_1 - \varepsilon\xi_1}{r}\dfrac{d\varphi_4}{dX_3} \\ \dots & \dots & \left(\dfrac{d\varphi_3}{dX_3} - \dfrac{x_1 - \varepsilon\xi_1}{r}\dfrac{d^2\varphi_1}{dX_3^2} - \dfrac{x_2 - \varepsilon\xi_2}{r}\dfrac{d^2\varphi_2}{dX_3^2}\right) \end{pmatrix} \qquad \text{in } \varepsilon\xi + B_1.$$

Then

$$\frac{\delta^2}{r}\mathcal{T}'_\varepsilon((\widetilde{\nabla v_\varepsilon})_S) \to \psi(x')\begin{pmatrix} 0 & 0 & -\dfrac{1}{2}X_2\dfrac{d\varphi_4}{dX_3} \\ \dots & 0 & \dfrac{1}{2}X_1\dfrac{d\varphi_4}{dX_3} \\ \dots & \dots & \dfrac{d\varphi_3}{dX_3} - X_1\dfrac{d^2\varphi_1}{dX_3^2} - X_2\dfrac{d^2\varphi_2}{dX_3^2} \end{pmatrix} = V(x', X) \quad \text{strongly in } L^2(\omega \times B_1).$$

Moreover,

$$\mathcal{T}'_\varepsilon(\widetilde{v}_\varepsilon) \to \psi(x')\begin{pmatrix} \varphi_1(X_3) \\ \varphi_2(X_3) \\ 0 \end{pmatrix} \qquad \text{strongly in } L^2(\omega \times B_1).$$

Unfolding the integral over Ω^i_ε we obtain

$$\begin{aligned}\int_{\Omega^i_\varepsilon} \sigma_\varepsilon : (\nabla v)_S dx &= \sum_{\xi \in \Xi_\varepsilon}\int_{\varepsilon\xi + B_{r,\delta}} \sigma_\varepsilon : (\widetilde{\nabla v_\varepsilon})_S dx \\ &= r^2\delta\sum_{\xi \in \Xi_\varepsilon}\int_{B_1}\mathcal{T}'_\varepsilon(\sigma_\varepsilon) : \mathcal{T}'_\varepsilon((\widetilde{\nabla v_\varepsilon})_S) dx'\, dX_1\, dX_2\, dX_3 \\ &= \frac{r^2\delta}{\varepsilon^2}\int_{\omega \times B_1}\mathcal{T}'_\varepsilon(\sigma_\varepsilon) : \mathcal{T}'_\varepsilon((\widetilde{\nabla v_\varepsilon})_S) dx'\, dX_1\, dX_2\, dX_3.\end{aligned}$$

In the same way for the integral for forces we get

$$\int_{\Omega^i_\varepsilon} f_\varepsilon \cdot v_\varepsilon dx = \frac{r^2\delta}{\varepsilon^2}\int_{\omega \times B_1}\mathcal{T}'_\varepsilon(f_\varepsilon) \cdot \mathcal{T}'_\varepsilon(v_\varepsilon) dx'\, dX_1\, dX_2\, dX_3.$$

Passing to the limit gives

$$\frac{\kappa_0^4}{\kappa_1^3}\int_{\omega \times B_1}\Theta : V dx'\, dX = \kappa_0^2\kappa_1\sum_{\alpha=1}^2\int_{\omega \times B_1} F^m_\alpha(x', X)\psi(x')\varphi_\alpha(X) dx'\, dX. \tag{6.3}$$

We can localize the above equation. That gives

$$\begin{aligned}\frac{\pi\kappa_0^2}{4\kappa_1^4}\mu^m\int_{\omega \times (0,1)}\frac{\partial\widetilde{\mathcal{R}}_3}{\partial X_3}\frac{d\varphi_4}{dX_3}\psi\, dx'\, dX_3 + \frac{\pi\kappa_0^2}{4\kappa_1^4}E^m\int_{\omega \times (0,1)}\left(4\frac{\partial\widetilde{\mathcal{U}}'_3}{\partial X_3}\frac{d\varphi_3}{dX_3} + \frac{\partial^2\widetilde{\mathcal{U}}_1}{\partial X_3^2}\frac{d^2\varphi_1}{dX_3^2} + \frac{\partial^2\widetilde{\mathcal{U}}_2}{\partial X_3^2}\frac{d^2\varphi_2}{dX_3^2}\right)\psi\, dx'\, dX_3 \\ = \int_{\omega \times (0,1)}\left(\widetilde{F}^m_1\varphi_1 + \widetilde{F}^m_2\varphi_2\right)\psi dx'\, dX_3.\end{aligned} \tag{6.4}$$

The density of the tensor product $\mathcal{C}_c^\infty(\omega)\otimes H_0^1(0,1)$ (resp. $\mathcal{C}_c^\infty(\omega)\otimes H_0^2(0,1)$) in $L^2(\omega;H_0^1(0,1))$ (resp. $L^2(\omega;H_0^2(0,1))$) implies

$$\begin{aligned}&\frac{\pi\kappa_0^2}{4\kappa_1^4}\mu^m\int_{\omega\times(0,1)}\frac{\partial\widetilde{\mathcal{R}}_3}{\partial X_3}\frac{\partial\Phi_4}{\partial X_3}\,dx'\,dX_3+\frac{\pi\kappa_0^2}{4\kappa_1^4}E^m\int_{\omega\times(0,1)}\Big(4\frac{\partial\widetilde{\mathcal{U}}_3'}{\partial X_3}\frac{\partial\Phi_3}{\partial X_3}+\frac{\partial^2\widetilde{\mathcal{U}}_1}{\partial X_3^2}\frac{\partial^2\Phi_1}{\partial X_3^2}+\frac{\partial^2\widetilde{\mathcal{U}}_2}{\partial X_3^2}\frac{\partial^2\Phi_2}{\partial X_3^2}\Big)\,dx'\,dX_3\\&=\int_{\omega\times(0,1)}\Big(\widetilde{F}_1^m\Phi_1+\widetilde{F}_2^m\Phi_2\Big)\,dx'\,dX_3\qquad\forall\Phi_3,\ \Phi_4\in L^2(\omega;H_0^1(0,1)),\quad\forall\Phi_1,\ \Phi_2\in L^2(\omega;H_0^2(0,1)).\end{aligned}\tag{6.5}$$

Step 2. Obtain $\widetilde{\mathcal{R}}_3,\widetilde{\mathcal{U}}_3'$.
Since $\varphi_3\in H_0^1(0,1)$ is not in the right-hand side of equation (6.4) we obtain

$$E^m\int_0^1\frac{\partial\widetilde{\mathcal{U}}_3'}{\partial X_3}\frac{d\varphi_3}{dX_3}dX_3=0\quad\Rightarrow\quad\frac{\partial^2\widetilde{\mathcal{U}}_3'}{\partial X_3^2}=0\quad\text{a.e. in}\quad\omega\times(0,1).\tag{6.6}$$

Moreover, we have $\widetilde{\mathcal{U}}_3'(x',0)=0$ for a.e. $x'\in\omega$. Therefore, there exists $a\in L^2(\omega)$ such that

$$\widetilde{\mathcal{U}}_3'(x',X_3)=X_3a(x'),\quad\text{for a.e.}\quad(x',X_3)\in\omega\times(0,1).$$

Similarly, recalling $\varphi_4\in H_0^1(0,1)$ and taking $\varphi_1=\varphi_2=\varphi_3=0$ in (6.4) lead to

$$\mu^m\int_0^1\frac{\partial\widetilde{\mathcal{R}}_3}{\partial X_3}\frac{d\varphi_4}{dX_3}dX_3=0\quad\Rightarrow\quad\frac{\partial^2\widetilde{\mathcal{R}}_3}{\partial X_3^2}=0\quad\text{a.e. in}\quad\omega\times(0,1),$$

which together with the boundary conditions (5.10) from Proposition 5.1 gives $\widetilde{\mathcal{R}}_3=0$. ☐

The variational problem (6.2) and the boundary conditions (5.15)-(5.16) allow to determine $\mathcal{U}_\alpha$ $(\alpha=1,2)$ in terms of the applied forces $\widetilde{F}_\alpha^m$ and the traces $u^\pm_{\alpha|\Sigma}$.

6.2 The equations for the macroscopic domain

Denote

$$\begin{aligned}\mathbb{V}=\Big\{&v\in L^2(\Omega^-\cup\Omega^+;\mathbb{R}^3)\mid v_{|\Omega^-}\in H^1(\Omega^-;\mathbb{R}^3)\ \text{and}\ v_{|\Omega^-}=0\ \text{on}\ \Gamma,\\&v_{|\Omega^+}\in H^1(\Omega^+;\mathbb{R}^3)\ \text{and}\ v_{3|\Omega^+}=v_{3|\Omega^-}\ \text{on}\ \Sigma\Big\}\\\mathbb{V}_T=\Big\{&(v,\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3,\mathcal{V}_4)\in\mathbb{V}\times[L^2(\Omega;H^2(0,1))]^2\times[L^2(\Omega;H^1(0,1))]^2\mid\\&\mathcal{V}_\alpha(\cdot,\cdot,0)=v^-_{\alpha\,|\Sigma},\quad\mathcal{V}_\alpha(\cdot,\cdot,1)=v^+_{\alpha\,|\Sigma}\ \text{a.e. in}\ \omega,\\&\mathcal{V}_3(\cdot,\cdot,0)=\mathcal{V}_4(\cdot,\cdot,0)=\mathcal{V}_4(\cdot,\cdot,1)=\frac{\partial\mathcal{V}_\alpha}{\partial X_3}(\cdot,\cdot,0)=\frac{\partial\mathcal{V}_\alpha}{\partial X_3}(\cdot,\cdot,1)=0\ \text{a.e. in}\ \omega,\quad\alpha\in\{1,2\}\Big\}\end{aligned}$$

Let χ be in $\mathcal{C}_c^\infty(\mathbb{R}^2)$ such that $\chi(y)=1$ in D_1 (the disc centered in $O=(0,0)$ and radius 1).

From now on we only consider the case (ii).

6.2.1 Determination of $\widetilde{\mathcal{U}}_3'$

Lemma 6.1. *The function a introduced in Proposition 6.1 is equal to 0 and*

$$\widetilde{\mathcal{U}}_3'(x',X_3)=0\qquad\text{a.e. in }\omega\times(0,1).$$

Proof. For any $\psi_3\in\mathcal{C}^1(\overline{\omega}\times[0,1])$ satisfying $\psi_3(x',0)=0$ for every $x'\in\omega$, we consider the following test function:

$$\begin{aligned}v_{\varepsilon,\alpha}(x)&=0\qquad\text{for a.e. }x\in\Omega_\varepsilon,\quad\alpha=1,2,\\v_\varepsilon(x)&=0\qquad\text{for a.e. }x\in\Omega^-,\\v_{\varepsilon,3}(x)&=\frac{r}{\delta}\Big[\psi_3(x',1)\Big(1-\chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\Big)+\psi_3\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y,1\Big)\chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\Big],\qquad\text{for a.e. }x\in\Omega_\varepsilon^+,\\v_{\varepsilon,3}(x)&=\frac{r}{\delta}\psi_3\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y,\frac{x_3}{\delta}\Big),\qquad\text{for a.e. }x\in\Omega_\varepsilon^i.\end{aligned}$$

If $\dfrac{r}{\varepsilon}$ is small enough, v_ε is an admissible test function. The symmetric strain tensor in Ω_ε^i is given by

$$(\nabla v_\varepsilon)_S = \frac{r}{\delta^2}\begin{pmatrix} 0 & 0 & 0 \\ \dots & 0 & 0 \\ \dots & \dots & \dfrac{\partial \psi_3}{\partial X_3}\Big(\varepsilon\xi, \dfrac{x_3}{\delta}\Big) \end{pmatrix} \qquad \text{a.e. in } \varepsilon\xi + B_{r,\delta}.$$

Then

$$\frac{\delta^2}{r}\mathcal{T}'_\varepsilon((\widetilde{\nabla v_\varepsilon})_S) \to \begin{pmatrix} 0 & 0 & 0 \\ \dots & 0 & 0 \\ \dots & \dots & \dfrac{\partial \psi_3}{\partial X_3}(x', X_3) \end{pmatrix} = V(x', X) \quad \text{ strongly in } L^2(\omega\times B_1).$$

Elements of the symmetric strain tensor in Ω_ε^+ are written as follows:

$$\begin{aligned}
&(\nabla v_\varepsilon)_S^{11} = (\nabla v_\varepsilon)_S^{22} = (\nabla v_\varepsilon)_S^{12} = (\nabla v_\varepsilon)_S^{33} = 0,\\
&(\nabla v_\varepsilon)_S^{\alpha 3} = (\nabla v_\varepsilon)_S^{3\alpha} = \frac{1}{2}\frac{r}{\delta}\frac{\partial \psi_3}{\partial x_\alpha}(x',1)(1-\chi(y)) + \frac{1}{2\delta}\frac{\partial \chi}{\partial y_\alpha}(y)\left(\psi_3(x',1) - \psi_3\left(\varepsilon\left[\frac{x'}{\varepsilon}\right]_Y, 1\right)\right),
\end{aligned}$$

where $y = \dfrac{\varepsilon}{r}\Big\{\dfrac{x'}{\varepsilon}\Big\}_Y$.

By Lemma 9.1 (see Appendix) and taking into account $\dfrac{r}{\delta} \to 0$, the following convergences hold:

$$\begin{aligned}
&v_\varepsilon(\cdot + \delta e_3) \longrightarrow 0 \quad \text{ strongly in } H^1(\Omega^+;\mathbb{R}^3),\\
&(\nabla v_\varepsilon)_S \longrightarrow 0 \quad \text{ strongly in } L^2(\Omega^+;\mathbb{R}^9).
\end{aligned}$$

Moreover,

$$\mathcal{T}'_\varepsilon(v_\varepsilon) \longrightarrow 0 \quad \text{ strongly in } H^1(\omega\times B_1;\mathbb{R}^3).$$

Using v_ε as a test function in (2.8) and passing to the limit in the unfolded formulation give

$$\int_{\omega\times(0,1)} \frac{\partial \widetilde{\mathcal{U}}'_3}{\partial X_3}(x', X_3)\frac{\partial \psi_3}{\partial X_3}(x', X_3)\, dx'\, dX = \int_{\omega\times(0,1)} a(x')\frac{\partial \psi_3}{\partial X_3}(x', X_3)\, dx'\, dX = 0.$$

Hence $a = 0$. Since the test functions are dense in

$$\mathbb{V}_s = \Big\{\Psi \in L^2(\omega; H^1(0,1)) \mid \Psi(x', 0) = 0 \text{ a.e. in } \omega\Big\}$$

we obtain

$$\int_{\omega\times(0,1)} \frac{\partial \widetilde{\mathcal{U}}'_3}{\partial X_3}(x', X_3)\frac{\partial \Psi}{\partial X_3}(x', X_3)\, dx'\, dX = 0 \qquad \forall \Psi \in \mathbb{V}_s. \tag{6.7}$$

□

As a consequence of the above Lemma and Proposition 6.1 one gets

$$\begin{aligned}
&\Theta_{ij} = 0, \qquad (i,j) \neq (3,3)\\
&\Theta_{33} = -E^m\Big(X_1\frac{\partial^2 \widetilde{\mathcal{U}}_1}{\partial X_3^2} + X_2\frac{\partial^2 \widetilde{\mathcal{U}}_2}{\partial X_3^2}\Big).
\end{aligned} \tag{6.8}$$

6.2.2 Determination of $u_\alpha^\pm$ and u_3

Theorem 6.1. *The variational formulation of the limit problem for* (2.8) *is*

$$\begin{aligned}
&\int_{\Omega^+\cup\Omega^-} \sigma^\pm : (\nabla v)_S\, dx + \frac{\pi\kappa_0^4}{4\kappa_1^3}E^m\int_{\omega\times(0,1)}\sum_{\alpha=1}^2 \frac{\partial^2 \widetilde{\mathcal{U}}_\alpha}{\partial X_3^2}\frac{\partial^2 \psi_\alpha}{\partial X_3^2}\, dx'\, dX_3\\
&+\frac{\pi\kappa_0^2}{4\kappa_1^4}\mu^m\int_{\omega\times(0,1)}\frac{\partial \widetilde{\mathcal{R}}_3}{\partial X_3}\frac{\partial \psi_4}{\partial X_3}\, dx'\, dX_3 + \frac{\pi\kappa_0^2}{\kappa_1^4}E^m\int_{\omega\times(0,1)}\frac{\partial \widetilde{\mathcal{U}}'_3}{\partial X_3}\frac{\partial \Phi_3}{\partial X_3}\, dx'\, dX_3\\
&= \int_{\Omega^+\cup\Omega^-} F\, v\, dx + \kappa_0^2\kappa_1\int_{\omega\times(0,1)}\sum_{\alpha=1}^2 \overline{F}_\alpha^m\psi_\alpha dx'\, dX_3 + \kappa_0^2\kappa_1\int_\omega \overline{F}_3^m v_3 dx',\\
&\qquad \forall (v, \psi_1, \psi_2, \psi_3, \psi_4) \in \mathbb{V}_T
\end{aligned} \tag{6.9}$$

where

$$\overline{F}_3^m(x') = \int_{B_1} F_3^m(x', X)\,dX, \quad x' \in \omega.$$

Proof. For any $v \in \mathbb{V}$ such that $v_{|\Omega^-} \in W^{1,\infty}(\Omega^-, \mathbb{R}^3)$ and $v_{|\Omega^+} \in W^{1,\infty}(\Omega^+, \mathbb{R}^3)$, we first define the displacement $v_{\varepsilon,r}$ in the following way:

$$v_{\varepsilon,r}(x) = v(x)\Big(1 - \chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\Big) + v\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y, x_3\Big)\chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big), \qquad \text{for a.e. } x \in \Omega^- \cup \Omega^+. \tag{6.10}$$

Then denote h the following function belonging to $W^{1,\infty}(-L, L)$

$$h(x_3) = \begin{cases} \dfrac{x_3 + L}{L}, & x_3 \in [-L, 0], \\ 1, & x_3 \geq 0. \end{cases} \tag{6.11}$$

Now consider the test displacement

$$\begin{aligned} &v'_\varepsilon(x) = v(x)\big(1 - h(x_3)\big) + v_{\varepsilon,r}(x)h(x_3), \qquad \text{for a.e. } x \in \Omega^-,\\ &v'_\varepsilon(x) = v_{\varepsilon,r}(x', x_3 - \delta), \qquad \text{for a.e. } x \in \Omega^+_\varepsilon,\\ &v'_\varepsilon(x) = \begin{pmatrix} \psi_1\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big]_Y, \dfrac{x_3}{\delta}\Big) \\ \psi_2\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big]_Y, \dfrac{x_3}{\delta}\Big) \\ v_3\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big]_Y, 0\Big) - \dfrac{\varepsilon}{\delta}\Big\{\dfrac{x'}{\varepsilon}\Big\}_Y \cdot \dfrac{\partial\psi}{\partial X_3}\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big]_Y, \dfrac{x_3}{\delta}\Big) \end{pmatrix} \qquad \text{for a.e. } x \in \Omega^i_\varepsilon, \end{aligned}$$

where $\psi_\alpha \in \mathcal{C}^1(\overline{\omega}; \mathcal{C}^3([0,1])), \alpha = 1, 2$, satisfies

$$\psi_\alpha(x', 0) = v_{\alpha|\Omega^-}(x', 0), \quad \psi_\alpha(x', 1) = v_{\alpha|\Omega^+}(x', 0) \qquad \text{for every } x' \in \omega.$$

If $\dfrac{r}{\varepsilon}$ is small enough, v'_ε is an admissible test displacement.

Then by Lemma 9.1 the following convergences hold:

$$\begin{aligned} &v'_\varepsilon(\cdot + \delta e_3) \longrightarrow v \quad \text{strongly in } H^1(\Omega^+; \mathbb{R}^3),\\ &v'_\varepsilon \longrightarrow v \quad \text{strongly in } H^1(\Omega^-; \mathbb{R}^3),\\ &(\nabla v'_\varepsilon)_S \longrightarrow (\nabla v)_S \quad \text{strongly in } L^2(\Omega^+ \cup \Omega^-; \mathbb{R}^9). \end{aligned}$$

Computation of the strain tensor in Ω^i_ε gives

$$\begin{aligned} &(\nabla v'_\varepsilon)^{ij}_S = 0 \qquad (i,j) \neq (3,3),\\ &(\nabla v'_\varepsilon)^{33}_S = -\frac{r}{\delta^2}\left(X_1 \frac{\partial^2\psi_1}{\partial X_3^2}\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y, X_3\Big) + X_2 \frac{\partial^2\psi_2}{\partial X_3^2}\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y, X_3\Big)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{T}'_\varepsilon(v'_\varepsilon) &\longrightarrow \begin{pmatrix} \psi_1(x', X_3) \\ \psi_2(x', X_3) \\ v_3(x', 0) \end{pmatrix} \qquad \text{strongly in } L^2(\omega \times B_1; \mathbb{R}^3),\\ \frac{\delta^2}{r}\mathcal{T}'_\varepsilon\Big((\nabla v'_\varepsilon)^{33}_S\Big) &\longrightarrow -\left(X_1 \frac{\partial^2\psi_1}{\partial X_3^2}(x', X_3) + X_2 \frac{\partial^2\psi_2}{\partial X_3^2}(x', X_3)\right) \qquad \text{strongly in } L^2(\omega \times B_1). \end{aligned}$$

Unfolding and passing to the limit in (2.8) give

$$\begin{aligned} \int_{\Omega^\pm} \sigma^\pm : (\nabla v)_S\,dx - \frac{\kappa_0^4}{\kappa_1^3}\int_{\omega\times B_1} \Theta : \left(X_1 \frac{\partial^2\psi_1}{\partial X_3^2} + X_2 \frac{\partial^2\psi_2}{\partial X_3^2}\right) dx'\,dX = \\ = \int_{\Omega^\pm} F\,v\,dx + \kappa_0^2\kappa_1 \int_{\omega\times B_1} (F_1^m\psi_1 + F_2^m\psi_2 + F_3 v_3)\,dx'\,dX. \end{aligned}$$

Since the space $W^{1,\infty}(\Omega^+; \mathbb{R}^3)$ is dense in $H^1(\Omega^+; \mathbb{R}^3)$, the space of functions in $W^{1,\infty}(\Omega^-, \mathbb{R}^3)$ vanishing on Γ is dense in $H^1(\Omega^-; \mathbb{R}^3)$ and the space $\mathcal{C}^1(\overline{\omega}; \mathcal{C}^3([0,1]))$ is dense in $L^2(\omega; H^1(0,1))$, the above equality holds for every v in $\mathbb{V}$ and every ψ_1, ψ_2 in $L^2(\omega; H^1(0,1))$ satisfying

$$\psi_\alpha(x', 0) = v_{\alpha|\Omega^-}(x', 0), \quad \psi_\alpha(x', 1) = v_{\alpha|\Omega^+}(x', 0) \qquad \text{for a.e. } x' \in \omega.$$

Finally, integrating over D_1 and due to (6.5), (6.7) and (6.8) we obtain the result. $\square$

6.2.3 The case (i)

We introduce the classical unfolding operator.

Definition 6.1. *For φ Lebesgue-measurable function on ω, the unfolding operator $\mathcal{T}''_\varepsilon$ is defined as follows:*

$$\mathcal{T}''_\varepsilon(\varphi)(s,y)=\begin{cases}\varphi\Big(\varepsilon\Big[\dfrac{s}{\varepsilon}\Big]_Y+\varepsilon y\Big), & \textit{for a.e.}\quad (s,y)\in\widehat{\omega}_\varepsilon\times Y,\\ 0, & \textit{for a.e.}\quad (s,y)\in\Lambda_\varepsilon\times Y.\end{cases}$$

Recall that (see [7])

Lemma 6.2. *Let ϕ be in $W^{1,\infty}(\omega)$ and ϕ_ε defined by*

$$\phi_\varepsilon(x')=\chi\Big(\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\phi\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y\Big)+\Big[1-\chi\Big(\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\Big]\phi(x')\quad \textit{for a.e.}\quad x'\in\omega.$$

Then we have

$$\begin{aligned}&\mathcal{T}''_\varepsilon(\phi_\varepsilon)\longrightarrow\phi\quad\textit{strongly in}\ \ L^2(\omega;H^1(Y)),\\ &\mathcal{T}''_\varepsilon(\nabla\phi_\varepsilon)\longrightarrow\nabla\phi\quad\textit{strongly in}\ \ L^2(\omega\times Y).\end{aligned}$$

Theorem 6.2. *The variational formulation for the problem (2.8) in the case (i)*

$$\begin{aligned}&\int_{\Omega^\pm}\sigma^\pm:(\nabla v)_S\,dx+\frac{\pi\kappa_0^4}{4\kappa_1^3}E^m\int_{\omega\times(0,1)}\sum_{\alpha=1}^2\frac{\partial^2\psi_\alpha}{\partial X_3^2}\frac{\partial^2\widetilde{\mathcal{U}}_\alpha}{\partial X_3^2}\,dx'dX_3=\\ &=\int_{\Omega^\pm}F\,v\,dx+\kappa_0^2\kappa_1\int_{\omega\times(0,1)}\sum_{\alpha=1}^2\widetilde{F}^m_\alpha\psi_\alpha dx'\,dX_3+\kappa_0^2\kappa_1\int_\omega\overline{F}^m_3 v_3dx',\\ &\forall v\in\mathbb{V},\qquad\forall\psi_1,\ \ \psi_2\in L^2(\omega;H^1(0,1))\quad\textit{satisfying}\\ &\psi_\alpha(x',0)=v_{\alpha|\Omega^-}(x',0),\quad \psi_\alpha(x',1)=v_{\alpha|\Omega^+}(x',0)\qquad\textit{for a.e. } x'\in\omega\end{aligned}\tag{6.12}$$

Proof. Step 1. Pass to the limit in the weak formulation.
To (5.7) and (5.8) we add

$$\mathcal{T}''_\varepsilon(u_\varepsilon)\rightharpoonup u^-\quad\text{weakly in } L^2(\Omega^-;H^1(Y)),\tag{6.13}$$

$$\mathcal{T}''_\varepsilon(\nabla u_\varepsilon)\rightharpoonup\nabla u^-+\nabla_y\widehat{u}^-\quad\text{weakly in } L^2(\Omega^-\times Y),\tag{6.14}$$

$$\mathcal{T}''_\varepsilon(u_\varepsilon)(\cdot+\delta e_3,\cdot\cdot)\rightharpoonup u^+\quad\text{weakly in } L^2(\Omega^+;H^1(Y)),\tag{6.15}$$

$$\mathcal{T}''_\varepsilon(\nabla u_\varepsilon)(\cdot+\delta e_3,\cdot\cdot)\rightharpoonup\nabla u^++\nabla_y\widehat{u}^+\quad\text{weakly in } L^2(\Omega^+\times Y),\tag{6.16}$$

where $\widehat{u}^-$ belongs to $L^2(\Omega^-;H^1_{per}(Y;\mathbb{R}^3))$ and $\widehat{u}^+$ belongs to $L^2(\Omega^+;H^1_{per}(Y;\mathbb{R}^3))$.

Remark 6.1. *Here the third variable of u_ε is considered as a parameter, on which the unfolding operator $\mathcal{T}''_\varepsilon$ does not have any effect.*

Step 2. Determination of $\mathcal{U}'_3$.

To determine the function a introduced in Proposition 6.1, take $\psi_3\in\mathcal{C}^1(\overline{\omega}\times[0,1])$ satisfying $\psi_3(x',0)=0$ for every $x'\in\omega$ and consider the following test function:

$$\begin{aligned}&v_{\varepsilon,\alpha}(x)=0\qquad\text{for a.e. } x\in\Omega_\varepsilon,\ \ \alpha=1,2,\\ &v_\varepsilon(x)=0\qquad\text{for a.e. } x\in\Omega^-,\\ &v_{\varepsilon,3}(x)=\varepsilon^{1/3}\Big[\psi_3(x',1)\Big(1-\chi\Big(\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\Big)+\psi_3\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y,1\Big)\chi\Big(\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\Big]\qquad\text{for a.e. } x\in\Omega^+_\varepsilon,\\ &v_{\varepsilon,3}(x)=\varepsilon^{1/3}\psi_3\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y,\frac{x_3}{\varepsilon^{2/3}}\Big)\chi\Big(\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\qquad\text{for a.e. } x\in\Omega^i_\varepsilon.\end{aligned}$$

We obtain the following convergences:

$$\begin{aligned}&v_\varepsilon(\cdot+\delta e_3)\longrightarrow 0\quad\text{strongly in } H^1(\Omega^+\cup\Omega^-;\mathbb{R}^3),\\ &(\nabla v_\varepsilon)_S\longrightarrow 0\quad\text{strongly in } L^2(\Omega^+\cup\Omega^-;\mathbb{R}^9),\\ &\mathcal{T}'_\varepsilon(v_\varepsilon)\longrightarrow 0\quad\text{strongly in } H^1(\omega\times B_1;\mathbb{R}^3).\end{aligned}$$

Unfolding and passing to the limit as in the Subsection 6.2.1 we obtain that $a=0$.

Step 3. For any $v \in \mathbb{V}$ such that $v_{|\Omega^-} \in W^{1,\infty}(\Omega^-;\mathbb{R}^3)$ and $v_{|\Omega^+} \in W^{1,\infty}(\Omega^+;\mathbb{R}^3)$, define the displacement v_ε in the following way:

$$v_\varepsilon(x) = v(x)\Big(1 - \chi\Big(\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\Big) + v\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y, x_3\Big)\chi\Big(\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big), \qquad \text{for a.e. } x \in \Omega^- \cup \Omega^+. \tag{6.17}$$

Consider the following test displacement:

$$v'_\varepsilon(x) = v(x)\big(1 - h(x_3)\big) + v_\varepsilon(x)h(x_3) + \varepsilon\Psi^{(-)}(x', x_3)\widehat{v}\left(\Big\{\frac{x'}{\varepsilon}\Big\}_Y\right), \qquad \text{for a.e. } x \in \Omega^-,$$

$$v'_\varepsilon(x) = v_\varepsilon(x', x_3 - \delta) + \varepsilon\Psi^{(+)}(x', x_3 - \delta)\widehat{v}\left(\Big\{\frac{x'}{\varepsilon}\Big\}_Y\right), \qquad \text{for a.e. } x \in \Omega^+_\varepsilon,$$

$$v'_\varepsilon(x) = \begin{pmatrix} \psi_1\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big]_Y, \dfrac{x_3}{\delta}\Big) \\ \psi_2\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big]_Y, \dfrac{x_3}{\delta}\Big) \\ v_3\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big]_Y, 0\Big) - \dfrac{\varepsilon}{\delta}\Big\{\dfrac{x'}{\varepsilon}\Big\}_Y \cdot \dfrac{\partial\psi}{\partial X_3}\Big(\varepsilon\Big[\dfrac{x'}{\varepsilon}\Big]_Y, \dfrac{x_3}{\delta}\Big) \end{pmatrix} \qquad \text{for a.e. } x \in \Omega^i_\varepsilon,$$

where

- $\widehat{v} \in H^1_{per}(Y;\mathbb{R}^3)$,
- $\psi_\alpha \in \mathcal{C}^1(\overline{\omega};\mathcal{C}^3([0,1])), \alpha = 1,2$, satisfies

$$\psi_\alpha(x', 0) = v_{\alpha|\Omega^-}(x', 0), \quad \psi_\alpha(x', 1) = v_{\alpha|\Omega^+}(x', 0) \qquad \text{for every } x' \in \omega,$$

- $\Psi^{(-)} \in W^{1,\infty}(\Omega^-)$, $\Psi^{(+)} \in W^{1,\infty}(\Omega^+)$ satisfying

$$\Psi^{(\pm)}(x', 0) = 0, \quad \text{a.e. in } \omega, \qquad \Psi^{(-)} = 0 \quad \text{on } \Gamma,$$

- $h(x_3)$ is defined as in (6.11).

Using (6.1) we obtain the following convergences:

$$\begin{aligned}
&\mathcal{T}''_\varepsilon(v'_\varepsilon(\cdot, \cdot\cdot)) \longrightarrow v \quad \text{strongly in } L^2(\Omega^-; H^1(Y)),\\
&\mathcal{T}''_\varepsilon(\nabla v'_\varepsilon(\cdot, \cdot\cdot)) \longrightarrow \nabla v + \Psi^{(-)}\nabla_y\widehat{v} \quad \text{strongly in } L^2(\Omega^- \times Y),\\
&\mathcal{T}''_\varepsilon(v'_\varepsilon(\cdot + \delta e_3, \cdot\cdot)) \longrightarrow v \quad \text{strongly in } L^2(\Omega^+; H^1(Y)),\\
&\mathcal{T}''_\varepsilon(\nabla v'_\varepsilon(\cdot + \delta e_3, \cdot\cdot)) \longrightarrow \nabla v + \Psi^{(+)}\nabla_y\widehat{v} \quad \text{strongly in } L^2(\Omega^+ \times Y).
\end{aligned}$$

Moreover,

$$\mathcal{T}_\varepsilon(\mathcal{T}''_\varepsilon(v'_\varepsilon)) \longrightarrow \begin{pmatrix} \psi_1(x', X_3) \\ \psi_2(x', X_3) \\ v_3(x', 0) \end{pmatrix} \quad \text{strongly in } L^2(\omega; H^1(Y \times B_1)),$$

$$\frac{\delta^2}{r}\mathcal{T}_\varepsilon\left(\mathcal{T}''_\varepsilon\left((\nabla v'_\varepsilon)^{33}_S\right)\right) \longrightarrow -X_1\frac{\partial^2\psi_1}{\partial X_3^2}(x', X_3) - X_2\frac{\partial^2\psi_2}{\partial X_3^2}(x', X_3) \quad \text{strongly in } L^2(\omega \times Y \times B_1).$$

Unfolding and passing to the limit we obtain

$$\begin{aligned}
\int_{\Omega^\pm \times Y} (\sigma^\pm + \widehat{\sigma}^\pm) : \Big((\nabla v)_S + \Psi^{(\pm)}(\nabla_y\widehat{v})_S\Big)\, dxdy - \frac{\kappa_0^4}{\kappa_1^3}\int_{\omega \times B_1} \Theta : \Big(X_1\frac{\partial^2\psi_1}{\partial X_3^2} + X_2\frac{\partial^2\psi_2}{\partial X_3^2}\Big)\, dx'\, dX = \\
= \int_{\Omega^\pm} F\, v\, dx + \kappa_0^2\kappa_1\int_{\omega \times B_1} (F_1^m\psi_1 + F_2^m\psi_2 + F_3 v_3)\, dx'\, dX.
\end{aligned} \tag{6.18}$$

Since $\sigma^\pm$ and $(\nabla v)_S$ do not depend on y and due to the periodicity of the fields $\widehat{v}$ and $\widehat{u}^\pm$, the above equality reads

$$\begin{aligned}
\int_{\Omega^\pm} \sigma^\pm : (\nabla v)_S\, dx + \int_{\Omega^\pm \times Y} \widehat{\sigma}^\pm : \Psi^{(\pm)}(\nabla_y\widehat{v})_S\, dxdy - \frac{\kappa_0^4}{\kappa_1^3}\int_{\omega \times B_1} \Theta : \Big(X_1\frac{\partial^2\psi_1}{\partial X_3^2} + X_2\frac{\partial^2\psi_2}{\partial X_3^2}\Big)\, dx'\, dX = \\
= \int_{\Omega^\pm} F\, v\, dx + \kappa_0^2\kappa_1\int_{\omega \times B_1} (F_1^m\psi_1 + F_2^m\psi_2 + F_3 v_3)\, dx'\, dX.
\end{aligned}$$

Step 3. To determine $\widehat{\sigma}$ we first take $v = 0$. We obtain

$$\int_{\Omega^\pm \times Y} \widehat{\sigma}^\pm : \Psi^{(\pm)}(\nabla_y \widehat{v})_S \, dxdy - \frac{\kappa_0^4}{\kappa_1^3} \int_{\omega \times B_1} \Theta : \left(X_1 \frac{\partial^2 \psi_1}{\partial X_3^2} + X_2 \frac{\partial^2 \psi_2}{\partial X_3^2} \right) dx' \, dX = \\ = \kappa_0^2 \kappa_1 \int_{\omega \times B_1} (F_1^m \psi_1 + F_2^m \psi_2) \, dx' \, dX.$$

Since the right-hand side does not contain $\widehat{v}$,

$$\int_{\Omega^\pm \times Y} \widehat{\sigma}^\pm : \Psi^{(\pm)}(\nabla_y \widehat{v})_S \, dxdy = 0,$$

which corresponds to the strong formulation

$$\begin{cases} \displaystyle\sum_{j=1}^3 \frac{\widehat{\sigma}_{ij}^\pm}{\partial y_j} = 0, & \text{in } \Omega^\pm \times Y, \\ \displaystyle\sum_{j=1}^3 \widehat{\sigma}_{ij}^\pm = 0, & \text{on } \partial(\Omega^\pm \times Y), \end{cases}$$

for $i = 1, 2, 3$. Therefore, $\widehat{\sigma}^\pm = 0$, and (6.18) is rewritten as

$$\int_{\Omega^\pm} \sigma^\pm : (\nabla v)_S \, dx - \frac{\kappa_0^4}{\kappa_1^3} \int_{\omega \times B_1} \Theta : \left(X_1 \frac{\partial^2 \psi_1}{\partial X_3^2} + X_2 \frac{\partial^2 \psi_2}{\partial X_3^2} \right) dx' \, dX = \\ = \int_{\Omega^\pm} F \, v \, dx + \kappa_0^2 \kappa_1 \int_{\omega \times B_1} (F_1^m \psi_1 + F_2^m \psi_2 + F_3 v_3) \, dx' \, dX. \quad (6.19)$$

Since the space $W^{1,\infty}(\Omega^+; \mathbb{R}^3)$ is dense in $H^1(\Omega^+; \mathbb{R}^3)$, the space of functions in $W^{1,\infty}(\Omega^-, \mathbb{R}^3)$ vanishing on Γ is dense in $H^1(\Omega^-; \mathbb{R}^3)$ and the space $\mathcal{C}^1(\overline{\omega}; \mathcal{C}^3([0,1]))$ is dense in $L^2(\omega; H^1(0,1))$, the above equality holds for every v in $\mathbb{V}$ and every ψ_1, ψ_2 in $L^2(\omega; H^1(0,1))$ satisfying

$$\psi_\alpha(x', 0) = v_{\alpha|\Omega^-}(x', 0), \quad \psi_\alpha(x', 1) = v_{\alpha|\Omega^+}(x', 0) \qquad \text{for a.e. } x' \in \omega.$$

Finally, integrating over D_1 and due to (6.8) we obtain the result. $\square$

7 Summarize

7.1 Strong formulation

Strong formulations are the same for the cases $(i), (ii)$. We will use the following notation.

Notation 7.1. *The convolution of the functions K and F is*

$$(K * \widetilde{F}_\alpha^m)(x', X_3) = \int_0^1 K(X_3, y_3) \widetilde{F}_\alpha^m(x', y_3) \, dy_3.$$

Let $\{\varepsilon\}$ be a sequence of positive real numbers which tends to 0. Let $(u_\varepsilon, \sigma_\varepsilon)$ be the solution of (2.8) and $\widetilde{\mathcal{U}}_\varepsilon$ and $\widetilde{\mathcal{R}}_\varepsilon$ be the two first terms of the decompostion of u_ε in Ω_ε^i. Let f satisfy assumptions (4.29). Then the limit problems for the cases $(i), (ii)$ can be written as follows.

Bending problem in the beams: $(\widetilde{\mathcal{U}}_1, \widetilde{\mathcal{U}}_2) \in L^2(\omega, H^1(0,1))^2$ is the unique solution of the problem

$$\begin{cases} \dfrac{\pi \kappa_0^2}{4 \kappa_1^4} E^m \dfrac{\partial^4 \widetilde{\mathcal{U}}_\alpha}{\partial X_3^4} = \widetilde{F}_\alpha^m & \text{a.e. in } \omega \times (0,1), \\ \dfrac{\partial \widetilde{\mathcal{U}}_\alpha}{\partial X_3}(\cdot, \cdot, 0) = \dfrac{\partial \widetilde{\mathcal{U}}_\alpha}{\partial X_3}(\cdot, \cdot, 1) = 0, & \text{a.e. in } \omega, \\ \widetilde{\mathcal{U}}_\alpha(\cdot, \cdot, 0) = u_{\alpha|\Sigma}^-, \qquad \widetilde{\mathcal{U}}_\alpha(\cdot, \cdot, 1) = u_{\alpha|\Sigma}^+, & \text{a.e. in } \omega, \end{cases} \quad (7.1)$$

3d elasticity problem in $\Omega^+ \cup \Omega^-$: $(u^\pm, \sigma^\pm) \in (H^1(\Omega^+ \cup \Omega^-))^3 \times (L^2(\Omega))_S^{3\times3}$ is the unique weak solution of the problem

$$-\sum_{j=1}^3 \frac{\partial \sigma_{ij}^\pm}{\partial x_j} = F_i \quad \text{in } \Omega^\pm, \quad i = 1, 2, 3,$$

together with the boundary conditions

$$\begin{cases} \sigma_{i3}^{+} = 0 & \text{in } \omega \times \{L\}, \\ \sigma_{i3}^{-} = 0 & \text{in } \omega \times \{-L\}, \end{cases}$$

and the transmission conditions

$$\begin{cases} [\sigma_{i3}^{\pm}]_{|\Sigma} = \overline{F}_i^m & \text{on } \Sigma, \\ [u_3^{\pm}]_{|\Sigma} = 0 & \text{on } \Sigma, \\ \sigma_{\alpha 3}^{+} = -\dfrac{3\pi\kappa_0^4}{\kappa_1^3} E^m [u_\alpha^{\pm}]_{|\Sigma} + \kappa_0^2\kappa_1 \displaystyle\int_0^1 K_\alpha * \widetilde{F}_\alpha^m dX_3 & \text{on } \Sigma, \quad \alpha = 1,2. \end{cases}$$

7.1.1 Derivation of the 3d problem

Lemma 7.1. *The weak formulation of the limit problem can be rewritten as*

$$\int_{\Omega^+ \cup \Omega^-} \sigma^{\pm} : (\nabla v)_S \, dx + \frac{3\pi\kappa_0^4}{\kappa_1^3} E^m \int_\Sigma \sum_{\alpha=1}^{2} [u_\alpha^{\pm}]_{|\Sigma} [v_\alpha^{\pm}]_{|\Sigma} \, ds =$$

$$= \int_{\Omega^+ \cup \Omega^-} F \, v \, dx + \kappa_0^2 \kappa_1 \int_\Sigma \sum_{\alpha=1}^{3} \overline{F}_\alpha^m \, v_\alpha^- \, ds + \kappa_0^2\kappa_1 \int_\Sigma \sum_{\alpha=1}^{2} [v_\alpha^{\pm}]_{|\Sigma} \int_0^1 K_\alpha * \widetilde{F}_\alpha^m dX_3 \, ds, \qquad \forall v \in \mathbb{V}, \quad (7.2)$$

where

$$\sigma^{\pm} = \lambda^b (\mathrm{Tr}\,(\nabla u^{\pm})_S) I + 2\mu^b (\nabla u^{\pm})_S \in L^2(\Omega^{\pm}; \mathbb{R}^9),$$

$$K_\alpha(X_3, y_3) = \delta(X_3 - y_3) X_3^2 (3 - 2X_3) + 6(1 - 2X_3)\big((X_3 - y_3) H(X_3 - y_3) + (1 - y_3)^2 (y_3 - 2y_3 X_3 - X_3)\big).$$

Proof. Step 1. Decomposition of $\widetilde{\mathcal{U}}_\alpha$.

Denote

$$\mathcal{V}_d = \Big\{ \eta \in \mathcal{C}^3([0,1]) \mid \eta(X_3) = (b-a) X_3^2 (3 - 2X_3) + a, \ \ (a,b) \in \mathbb{R}^2 \Big\}.$$

Observe that a function $X_3 \longmapsto \eta(X_3) = (b-a)X_3^2(3-2X_3) + a$ of $\mathcal{V}_d$ satisfies

$$\eta(0) = a, \quad \eta(1) = b, \quad \frac{d\eta}{dX_3}(0) = 0, \quad \frac{d\eta}{dX_3}(1) = 0, \quad \text{and} \quad \frac{d^4\eta}{dX_3^4} = 0 \ \text{in} \ (0,1).$$

Hence for any function $\psi \in H_0^2(0,1)$ we have

$$\int_0^1 \frac{d^2\eta}{dX_3^2}(t) \frac{d^2\psi}{dX_3^2}(t) \, dt = 0.$$

Let $\widetilde{\widetilde{\mathcal{U}}}_\alpha$ be in $L^2(\omega; H_0^2(0,1))$ the solution of the following problem:

$$\begin{cases} \dfrac{\pi\kappa_0^2}{4\kappa_1^4} E^m \dfrac{\partial^4 \widetilde{\widetilde{\mathcal{U}}}_\alpha}{\partial X_3^4}(x', X_3) = \widetilde{F}_\alpha^m(x', X_3) & \text{a.e. in } \omega \times (0,1), \\ \dfrac{\partial \widetilde{\widetilde{\mathcal{U}}}_\alpha}{\partial X_3}(\cdot,\cdot,0) = \dfrac{\partial \widetilde{\widetilde{\mathcal{U}}}_\alpha}{\partial X_3}(\cdot,\cdot,1) = 0, \quad \text{a.e. in } \omega, \\ \widetilde{\widetilde{\mathcal{U}}}_\alpha(\cdot,\cdot,0) = \widetilde{\widetilde{\mathcal{U}}}_\alpha(\cdot,\cdot,1) = 0, \quad \text{a.e. in } \omega. \end{cases}$$

Using Green's function we can write $\widetilde{\widetilde{\mathcal{U}}}_\alpha$ in the following way:

$$\widetilde{\widetilde{\mathcal{U}}}_\alpha(x', X_3) = \frac{4\kappa_1^4}{\pi E^m \kappa_0^2} \int_0^1 \xi_\alpha(X_3, y_3) \widetilde{F}_\alpha^m(x', y_3) \, dy_3,$$

where ξ_α is the solution of the equation

$$\begin{cases} \dfrac{d^4 \xi_\alpha}{dX_3^4} = \delta(X_3 - y_3), \quad y_3 \in (0,1), \\ \dfrac{d\xi_\alpha}{dX_3}(0) = \dfrac{d\xi_\alpha}{dX_3}(1) = 0, \\ \xi_\alpha(0) = \xi_\alpha(1) = 0. \end{cases}$$

Solving the above equation we obtain

$$\xi_\alpha(X_3,y_3)=\frac{1}{6}(X_3-y_3)^3H(X_3-y_3)-\frac{1}{6}(1-y_3)^2(2y_3+1)X_3^3+\frac{1}{2}(1-y_3)^2y_3X_3^2,$$

where H is the Heaviside function.

The function $\widetilde{\mathcal{U}}_\alpha$ is uniquely decomposed as a function belonging to $L^2(\omega;\mathcal{V}_d)$ and a function in $L^2(\omega;H_0^2(0,1))$

$$\begin{aligned}\widetilde{\mathcal{U}}_\alpha(x',X_3)&=(1-X_3)^2(2X_3+1)u^-_{\alpha|\Sigma}(x')-X_3^2(3-2X_3)u^+_{\alpha|\Sigma}(x')+\widetilde{\widetilde{\mathcal{U}}}_\alpha(x',X_3)\\&=\overline{\mathcal{U}}_\alpha(x',X_3)+\widetilde{\widetilde{\mathcal{U}}}_\alpha(x',X_3)\qquad\text{for a.e. }(x',X_3)\in\omega\times(0,1).\end{aligned}\tag{7.3}$$

Step 2. Taking into account decomposition (7.3) and using as a test function $\psi_\alpha=[v_\alpha^\pm]_{|\Sigma}X_3^2(3-2X_3)+v^-_{\alpha|\Sigma}$ in (6.9) we obtain

$$\begin{aligned}\int_{\Omega^+\cup\Omega^-}\sigma^\pm:(\nabla v)_S\,dx+\frac{3\pi\kappa_0^4}{2\kappa_1^3}E^m\int_\omega\sum_{\alpha=1}^2\int_0^1\Big(\frac{\partial^2\overline{\mathcal{U}}_1}{\partial X_3^2}[v_1^\pm]_{|\Sigma}+\frac{\partial^2\overline{\mathcal{U}}_2}{\partial X_3^2}[v_2^\pm]_{|\Sigma}\Big)(1-2X_3)\,dX_3\,dx'=\\=\int_{\Omega^+\cup\Omega^-}F\,v\,dx+\int_\omega\sum_{\alpha=1}^2[v_\alpha^\pm]_{|\Sigma}\int_0^1\Big(\kappa_0^2\kappa_1\widetilde{F}^m_\alpha X_3^2(3-2X_3)-\frac{3\pi\kappa_0^4}{2\kappa_1^3}E^m\frac{\partial^2\widetilde{\widetilde{\mathcal{U}}}_\alpha}{\partial X_3^2}(1-2X_3)\Big)dX_3\,dx'+\\+\kappa_0^2\kappa_1\int_\omega\sum_{\alpha=1}^3\overline{F}^m_\alpha\,v_\alpha^-\,dx'.\end{aligned}\tag{7.4}$$

Making use of the solutions for $\overline{\mathcal{U}}_\alpha$ and $\widetilde{\widetilde{\mathcal{U}}}_\alpha$ we can write

$$\begin{aligned}\int_{\Omega^+\cup\Omega^-}\sigma^\pm:(\nabla v)_S\,dx+\frac{3\pi\kappa_0^4}{\kappa_1^3}E^m\int_\omega\sum_{\alpha=1}^2[u_\alpha^\pm]_{|\Sigma}[v_\alpha^\pm]_{|\Sigma}\,dx'=\int_{\Omega^+\cup\Omega^-}F\,v\,dx+\\+\int_\omega\sum_{\alpha=1}^2[v_\alpha^\pm]_{|\Sigma}\int_0^1\kappa_0^2\kappa_1\Big(\widetilde{F}^m_\alpha X_3^2(3-2X_3)-6(1-2X_3)\int_0^1\frac{d^2\xi_\alpha}{dX_3^2}(X_3,y_3)\widetilde{F}^m_\alpha(x',y_3)dy_3\Big)dX_3\,dx'+\\+\kappa_0^2\kappa_1\int_\omega\sum_{\alpha=1}^3\overline{F}^m_\alpha\,v_\alpha^-\,dx'.\end{aligned}\tag{7.5}$$

Using the notation for convolution and the expression for $\dfrac{d^2\xi_\alpha}{dX_3^2}$ we get the result. $\square$

From variational formulation (7.2) the final strong formulation is obtained.

7.2 Convergences

Theorem 7.1. *Under the assumptions (4.29) on the applied forces, we first have (convergence of the stress energy)*

$$\begin{aligned}\lim_{\varepsilon\to0}\int_{\Omega_{r,\varepsilon,\delta}}\sigma_{r,\varepsilon,\delta}:(\nabla u_\varepsilon)_Sdx=\int_{\Omega^+\cup\Omega^-}\sigma^\pm:(\nabla u)_S\,dx+\frac{\pi\kappa_0^4}{4\kappa_1^3}E^m\int_{\omega\times(0,1)}\sum_{\alpha=1}^2\Big|\frac{\partial^2\widetilde{\mathcal{U}}_\alpha}{\partial X_3^2}\Big|^2dx'\,dX_3\\+\frac{\pi\kappa_0^2}{4\kappa_1^4}\mu^m\int_{\omega\times(0,1)}\Big|\frac{\partial\widetilde{\mathcal{R}}_3}{\partial X_3}\Big|^2dx'\,dX_3+\frac{\pi\kappa_0^2}{\kappa_1^4}E^m\int_{\omega\times(0,1)}\Big|\frac{\partial\widetilde{\mathcal{U}}_3'}{\partial X_3}\Big|^2dx'\,dX_3.\end{aligned}\tag{7.6}$$

The sequence $(u_\varepsilon,\sigma_\varepsilon)$ satisfy the following convergences:

- $u_\varepsilon\to u^-$ *strongly in* $H^1(\Omega^-)$,
  $u_\varepsilon(\cdot+\delta e_3)\to u^+$ *strongly in* $H^1(\Omega^+)$,
- $\sigma_\varepsilon\to\sigma^-$ *strongly in* $L^2(\Omega^-)$,
  $\sigma_\varepsilon(\cdot+\delta e_3)\to\sigma^+$ *strongly in* $L^2(\Omega^+)$,
- $\mathcal{T}'_\varepsilon(u_{\varepsilon,\alpha})\to\widetilde{\mathcal{U}}_\alpha$ *strongly in* $H^1(\omega\times(0,1))$, *where* $\widetilde{\mathcal{U}}_\alpha$ *is the solution of* (7.1), $\quad\alpha=1,2$,
  $\mathcal{T}'_\varepsilon(u_{\varepsilon,3})\to u_3^\pm(x',0)$ *strongly in* $H^1(\omega\times(0,1))$,

- $\dfrac{\delta^2}{r}\mathcal{T}'_\varepsilon(\sigma_\varepsilon) \to \Theta$ *strongly in* $L^2(\omega\times B_1)$, *where*

$$\Theta_{ij} = \begin{cases} -E^m\Big(X_1\dfrac{\partial^2\widetilde{\mathcal{U}}_1}{\partial X_3^2} + X_2\dfrac{\partial^2\widetilde{\mathcal{U}}_2}{\partial X_3^2}\Big), & (i,j)=(3,3),\\ 0, & \textit{otherwise}.\end{cases}$$

Proof. Step 1. We prove (7.6).

We first recall the classical identity: if T is a symmetric 3×3 matrix we have

$$\begin{aligned}\lambda^m Tr(T)Tr(T) + \sum_{i,j=1}^{3} 2\mu^m T_{ij}T_{ij} = E^m T_{33}^2 + \frac{E^m}{(1+\nu^m)(1-2\nu^m)}(T_{11}+T_{22}+2\nu^m T_{33})^2\\ + \frac{E^m}{2(1+\nu^m)}[(T_{11}-T_{22})^2 + 4(T_{12}^2+T_{13}^2+T_{23}^2)].\end{aligned}$$

Taking $\varphi = u_\varepsilon$ in (2.8), by standard weak lower-semi-continuity we obtain

[Right hand side of (7.6) (written with the Lamé's constants)] $\le \liminf\limits_{\varepsilon\to 0}$[left hand side of (7.6)].

Then we prove the inequality with $\limsup$ thanks to the variational formulation (6.9).

The equality (7.6) implies the strong converge of the stress and strain tensor fields. then we deduce the strong convergence of the different components of the displacement field.

Convergences in the domains $\Omega^\pm$.

From (5.1) we have, that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u^- \quad \text{weakly in } H^1(\Omega^-),\ \text{strongly in } L^2(\Omega^-),\\ u_\varepsilon(\cdot+\delta e_3) &\rightharpoonup u^+ \quad \text{weakly in } H^1(\Omega^+),\ \text{strongly in } L^2(\Omega^+).\end{aligned}$$

Therefore, we should prove the convergence of the gradients. Consider the domain Ω^- (the procedure for the domain Ω^+ is similar). Estimating the norm of the difference due to the coercivity we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\|\nabla u_\varepsilon - \nabla u^-\|^2_{L^2(\Omega^-)} &\le \alpha\lim_{\varepsilon\to 0}\int_{\Omega^-}(\sigma_\varepsilon-\sigma^-):((\nabla u_\varepsilon)_S - (\nabla u^-)_S)\,dx =\\ &= \alpha\lim_{\varepsilon\to 0}\int_{\Omega^-}\big(\sigma_\varepsilon:(\nabla u_\varepsilon)_S + \sigma^-:(\nabla u^-)_S - \sigma^-:(\nabla u_\varepsilon)_S - \sigma_\varepsilon^-:(\nabla u^-)_S\big)\,dx. \qquad (7.7)\end{aligned}$$

For the first term of the sum we get

$$\lim_{\varepsilon\to 0}\int_{\Omega^-}\sigma_\varepsilon:(\nabla u_\varepsilon)_S\,dx = \lim_{\varepsilon\to 0}\int_{\Omega^-} f u_\varepsilon\,dx = \int_{\Omega^-} f u^-\,dx.$$

Moreover,

$$-\lim_{\varepsilon\to 0}\int_{\Omega^-}\big(\sigma_\varepsilon:(\nabla u^-)_S + \sigma^-:(\nabla u_\varepsilon)_S\big)\,dx = -2\lim_{\varepsilon\to 0}\int_{\Omega^-}\sigma_\varepsilon:(\nabla u^-)_S\,dx = -2\int_{\Omega^-}\sigma^-:(\nabla u^-)_S\,dx.$$

Returning to (7.7) due to the variational formulation of the problem we derive

$$\frac{1}{\alpha}\lim_{\varepsilon\to 0}\|\nabla u_\varepsilon - \nabla u^-\|^2_{L^2(\Omega^-)} \le \int_{\Omega^-} f u^-\,dx - 2\int_{\Omega^-}\sigma^-:(\nabla u^-)_S\,dx + \int_{\Omega^-}\sigma^-:(\nabla u^-)_S\,dx = 0.$$

Then, the first 4 convergences are true.

Step 2. Convergences in the beams.

For $(s,X_1,X_2,X_3)\in(\omega\times D_1\times(0,1))$ we can write the following:

$$\mathcal{T}'_\varepsilon(u_{\varepsilon,i}) - \widetilde{\mathcal{U}}_i = \widetilde{\mathcal{U}}_{\varepsilon,i}(s,\delta X_3) + \widetilde{\mathcal{R}}_{\varepsilon,i}(s,\delta X_3)\wedge(rX_1e_1 + rX_2e_2) + \widetilde{\overline{u}}_{\varepsilon,i}(s,rX_1,rX_2,\delta X_3) - \widetilde{\mathcal{U}}_i(s,X_3), \quad i=1,2,3.$$

Based on the results of Proposition 5.1 we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\|\widetilde{\mathcal{U}}_{\varepsilon,i}(s,\delta X_3) + \widetilde{\mathcal{R}}_{\varepsilon,i}(s,\delta X_3)\wedge(rX_1e_1+rX_2e_2) + \widetilde{\overline{u}}_{\varepsilon,i}(s,rX_1,rX_2,\delta X_3) - \widetilde{\mathcal{U}}_i(s,X_3)\|^2_{L^2(\omega\times B_1)} \le\\ \le \lim_{\varepsilon\to 0}\|\widetilde{\mathcal{U}}_{\varepsilon,i}(s,\delta X_3) - \widetilde{\mathcal{U}}_i(s,X_3)\|^2_{L^2(\omega\times B_1)} + \lim_{\varepsilon\to 0}\|\widetilde{\mathcal{R}}_{\varepsilon,i}(s,\delta X_3)\wedge(rX_1e_1+rX_2e_2)\|^2_{L^2(\omega\times B_1)} +\\ + \lim_{\varepsilon\to 0}\|\widetilde{\overline{u}}_{\varepsilon,i}(s,rX_1,rX_2,\delta X_3)\|^2_{L^2(\omega\times B_1)} = 0.\end{aligned}$$

Consider the elements of the gradient:

$$\frac{\partial}{\partial X_j}\Big(\widetilde{\mathcal{U}}_{\varepsilon,i}(s,\delta X_3)+\widetilde{\mathcal{R}}_{\varepsilon,i}(s,\delta X_3)\wedge(rX_1e_1+rX_2e_2)+\widetilde{\widetilde{u}}_{\varepsilon,\alpha}(s,rX_1,rX_2,\delta X_3)-\widetilde{\mathcal{U}}_i(s,X_3)\Big),\quad i,j=1,2,3.$$

Again, based on the results of Proposition 5.1,

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial X_j}\Big(r\widetilde{\mathcal{R}}_{\varepsilon,i}(s,\delta X_3)\wedge(X_1e_1+X_2e_2)+\widetilde{\widetilde{u}}_{\varepsilon,i}(s,rX_1,rX_2,\delta X_3)\Big)=0,$$

and

$$\lim_{\varepsilon\to 0}\Big\|\frac{\partial}{\partial X_j}\Big(\widetilde{\mathcal{U}}_{\varepsilon,i}(s,\delta X_3)-\widetilde{\mathcal{U}}_i(s,X_3)\Big)\Big\|^2_{L^2(\omega\times(0,1))}=0,$$

which implies the convergences 5 and 6.

For the convergences of stress tensor in the beams we have

$$\begin{aligned}\Big\|\frac{\delta^2}{r}\mathcal{T}'_\varepsilon(\sigma_\varepsilon)-\Theta\Big\|^2_{L^2(\omega\times B_1)}&=\int_{\omega\times B_1}\Big(\frac{\delta^2}{r}\mathcal{T}'_\varepsilon(\sigma_\varepsilon)-\Theta\Big):\Big(\frac{\delta^2}{r}\mathcal{T}'_\varepsilon(\sigma_\varepsilon)-\Theta\Big)\,dx'\,dX=\\&=\frac{\delta^4}{r^2}\int_{\omega\times B_1}\mathcal{T}'_\varepsilon(\sigma_\varepsilon):\mathcal{T}'_\varepsilon(\sigma_\varepsilon)\,dx'\,dX-2\frac{\delta^2}{r}\int_{\omega\times B_1}\mathcal{T}'_\varepsilon(\sigma_\varepsilon):\Theta\,dx'\,dX+\int_{\omega\times B_1}\Theta:\Theta\,dx'\,dX.\end{aligned}$$

Passing to the limit we derive

$$\lim_{\varepsilon\to 0}\Big\|\frac{\delta^2}{r}\mathcal{T}'_\varepsilon(\sigma_\varepsilon)-\Theta\Big\|^2_{L^2(\omega\times B_1)}=\frac{\delta^4}{r^2}\int_{\omega\times B_1}\mathcal{T}'_\varepsilon(\sigma_\varepsilon):\mathcal{T}'_\varepsilon(\sigma_\varepsilon)\,dx'\,dX-\int_{\omega\times B_1}\Theta:\Theta\,dx'\,dX\leq ...$$

□

8 Complements

Remark 8.1. *The case*

$$r=\kappa_1\varepsilon^2,\quad \delta=\kappa_2\varepsilon^2,\quad \kappa_1,\kappa_2>0,$$

can also be considered, but should be studied separately. The structure obtained will no longer correspond to the set of the thin beams but to some kind of the perforated domain.

Remark 8.2. *For the case* $\dfrac{\varepsilon^2\delta^3}{r^4}\to 0$ *from the estimates (4.19), (4.20) we obtain, that*

$$\lim_{r,\varepsilon,\delta\to 0}\|u(\cdot,\cdot,\delta)-u(\cdot,\cdot,0)\|_{L^2(\widehat{\omega}_\varepsilon)}=0.$$

Therefore,

$$u^+{}_{|\Sigma}=u^-{}_{|\Sigma},$$

where $u^\pm\in H^1(\Omega^+\cup\Omega^-,\Gamma)$ *is the limit of the function* u_ε*. Hence we obtain two limit problems on the domains* Ω^+*,* Ω^- *with Dirichlet boundary conditions and the layer has no influence on the limit problem.*

9 Appendix

Let χ be in $\mathcal{C}_c^\infty(\mathbb{R}^2)$ such that $\chi(y)=1$ in D_1.

Lemma 9.1. *Let* ϕ *be in* $W^{1,\infty}(\omega)$ *and* $\phi_{\varepsilon,r}$ *defined by*

$$\phi_{\varepsilon,r}(x')=\chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\phi\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y\Big)+\Big[1-\chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\Big]\phi(x')\quad\text{for a.e. } x'\in\omega.$$

If $\dfrac{r}{\varepsilon}\to 0$ *then for every* $p\in[1,+\infty)$ *we have*

$$\phi_{\varepsilon,r}\longrightarrow\phi\quad\text{strongly in } W^{1,p}(\omega).$$

Proof. For the sake of simplicity we extend ϕ in a function belonging to $W^{1,\infty}(\mathbb{R}^2)$ still denoted ϕ. We denote

$$\widetilde{\Xi}_\varepsilon = \Big\{\xi \in \mathbb{Z}^2 \; ; \; (\varepsilon\xi + \varepsilon Y) \cap \omega \neq \emptyset\Big\}.$$

Observe that $\Xi_\varepsilon \subset \widetilde{\Xi}_\varepsilon$. Consider the following estimate:

$$\begin{aligned} \|\phi_{\varepsilon,r} - \phi\|_{L^\infty(\omega)} &= \Big\|\chi\Big(\frac{\varepsilon}{r}\Big\{\frac{\cdot}{\varepsilon}\Big\}_Y\Big)\Big(\phi\Big(\varepsilon\Big[\frac{\cdot}{\varepsilon}\Big]_Y\Big) - \phi\Big)\Big\|_{L^\infty(\omega)} \le \sup_{\xi\in\widetilde{\Xi}_\varepsilon}\Big\|\chi\Big(\frac{\cdot}{r}\Big)(\phi(\varepsilon\xi) - \phi(\varepsilon\xi + \cdot))\Big\|_{L^\infty(Y_\varepsilon)} \\ &= \sup_{\xi\in\widetilde{\Xi}_\varepsilon}\Big\|\chi\Big(\frac{\varepsilon}{r}\cdot\Big)(\phi(\varepsilon\xi) - \phi(\varepsilon\xi + \varepsilon\cdot))\Big\|_{L^\infty(Y)} \le \varepsilon\|\chi\|_{L^\infty(\mathbb{R}^2)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}. \end{aligned} \tag{9.1}$$

The partial derivative of $\phi_{\varepsilon,r} - \phi$ with respect to x_α is

$$\begin{aligned} &\frac{\partial(\phi_{\varepsilon,r} - \phi)}{\partial x_\alpha}(x') = \frac{1}{r}\frac{\partial\chi}{\partial X_\alpha}\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\Big(\phi\Big(\varepsilon\Big[\frac{x'}{\varepsilon}\Big]_Y\Big) - \phi(x')\Big) - \chi\Big(\frac{\varepsilon}{r}\Big\{\frac{x'}{\varepsilon}\Big\}_Y\Big)\frac{\partial\phi}{\partial x_\alpha}(x'), \quad \text{for a.e. } x' \in \omega, \\ &\frac{\partial(\phi_{\varepsilon,r} - \phi)}{\partial x_\alpha}(\varepsilon\xi + \varepsilon y') = \frac{1}{r}\frac{\partial\chi}{\partial X_\alpha}\Big(\frac{\varepsilon}{r}y'\Big)(\phi(\varepsilon\xi) - \phi(\varepsilon\xi + \varepsilon y')) - \chi\Big(\frac{\varepsilon}{r}y'\Big)\frac{\partial\phi}{\partial x_\alpha}(\varepsilon\xi + \varepsilon y'), \quad \xi \in \widetilde{\Xi}_\varepsilon, \text{ for a.e. } y' \in Y. \end{aligned}$$

Since χ has a compact support in $\mathbb{R}^2$, there exists $R > 0$ such that $\text{supp}(\chi) \subset D_R$. Thus, the support of the function $y' \longmapsto \chi\Big(\frac{\varepsilon}{r}y'\Big)$ is included in the disc $D_{rR/\varepsilon}$. As a consequence we get for a.e. $y' \in D_{rR/\varepsilon}$

$$|\phi(\varepsilon\xi) - \phi(\varepsilon\xi + \varepsilon y')| \le rR\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}.$$

Using the above estimate, for the norms of the derivatives we first have

$$\begin{aligned} \Big\|\frac{\partial(\phi_{\varepsilon,r} - \phi)}{\partial x_\alpha}\Big\|^p_{L^p(\varepsilon\xi+\varepsilon Y)} &= \varepsilon^2\Big\|\frac{\partial\chi}{\partial X_\alpha}\Big(\frac{\varepsilon}{r}\cdot\Big)\frac{\phi(\varepsilon\xi) - \phi(\varepsilon\xi + \varepsilon\cdot)}{r} - \chi\Big(\frac{\varepsilon}{r}\cdot\Big)\frac{\partial\phi}{\partial x_\alpha}(\varepsilon\xi + \varepsilon\cdot)\Big\|^p_{L^p(Y)} \\ &\le Cr^2\|\nabla\chi\|^p_{L^\infty(\mathbb{R}^2)}\|\nabla\phi\|^p_{L^\infty(\mathbb{R}^2)}. \end{aligned}$$

The constant does not depend on ε and r. Combining the above estimates for $\xi \in \widetilde{\Xi}_\varepsilon$, that gives

$$\|\nabla(\phi_{\varepsilon,r} - \phi)\|_{L^p(\omega)} \le C\Big(\frac{r}{\varepsilon}\Big)^{2/p}\|\nabla\chi\|_{L^\infty(\mathbb{R}^2)}\|\nabla\phi\|_{L^\infty(\mathbb{R}^2)}. \tag{9.2}$$

The constant does not depend on r and ε. Hence, estimates (9.1) and (9.2) imply that ϕ_ε strongly converges toward ϕ in $W^{1,p}(\omega)$. ☐

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